On Packing Colorings of Distance Graphs

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The *packing chromatic number* $\chi_\rho(G)$ of a graph G is the least integer k for which there exists a mapping f from $V(G)$ to $\{1, 2, \ldots, k\}$ such that any two vertices of color i are at a distance of at least $i+1$. This paper studies the packing chromatic number of infinite distance graphs $G(\mathbb{Z}, D)$, i.e. graphs with the set $\mathbb{Z}$ of integers as vertex set, with two distinct vertices $i, j \in \mathbb{Z}$ being adjacent if and only if $|i - j| \in D$. We present lower and upper bounds for $\chi_\rho(G(\mathbb{Z}, D))$, showing that for finite D, the packing chromatic number is finite. Our main result concerns distance graphs with $D = \{1, t\}$ for which we prove some upper bounds on their packing chromatic numbers, the smaller ones being for $t \geq 447$: $\chi_\rho(G(\mathbb{Z}, \{1, t\})) \leq 40$ if t is odd and $\chi_\rho(G(\mathbb{Z}, \{1, t\})) \leq 81$ if t is even.

Keywords: graph coloring; packing chromatic number; distance graph.

1. Introduction

Let G be a connected graph and let k be an integer, $k \geq 1$. A *packing k-coloring* (or simply a packing coloring) of a graph G is a mapping f from $V(G)$ to $\{1, 2, \cdots, k\}$ such that for any two distinct vertices u and v, if $f(u) = f(v) = i$, then $\mathrm{dist}(u, v) > i$, where $\mathrm{dist}(u, v)$ is the distance between u and v in G (thus vertices of color i form an i-packing of G). The *packing chromatic number* $\chi_\rho(G)$ of G is the smallest integer k for which G has a packing k-coloring.

This parameter was introduced recently by Goddard et al. [9] under the name of *broadcast chromatic number* and the authors showed that deciding whether $\chi_\rho(G) \leq 4$ is NP-hard. Fiala and Golovach [6] showed that the packing coloring problem is NP-complete for trees. Brešar et al. [2] studied the problem on Cartesian products graphs, hexagonal lattice and trees, using the name of packing chromatic number. Other studies on this parameter mainly concern infinite graphs, with a natural question to be answered : *does a given infinite graph have finite packing chromatic number ?* Goddard et al. answered this question affirmatively for the infinite two dimensional square grid by showing $9 \leq \chi_\rho \leq 23$. The lower bound was later improved to 10 by Fiala et al. [7] and then to 12 by Ekstein et al. [5]. The upper bound

was recently improved by Holub and Soukal [13] to 17. Fiala et al. [7] showed that the infinite hexagonal grid has packing chromatic number 7; while both the infinite triangular lattice and the 3-dimensional square lattice were shown to admit no finite packing coloring by Finbow and Rall [8]. Infinite product graphs were considered by Fiala et al. [7] who showed that the product of a finite path (of order at least two) with the 2-dimensional square grid has infinite packing chromatic number while the product of the infinite path and any finite graph has finite packing chromatic number.

The (infinite) *distance graph* $G(\mathbb{Z}, D)$ with distance set $D = \{d_1, d_2, \ldots, d_k\}$, where d_i are positive integers, has the set $\mathbb{Z}$ of integers as vertex set, with two distinct vertices $i, j \in \mathbb{Z}$ being adjacent if and only if $|i - j| \in D$. The *finite distance graph* $G_n(D)$ is the subgraph of $G(\mathbb{Z}, D)$ induced by vertices $0, 1, \ldots, n-1$. To simplify, $G(\mathbb{Z}, \{d_1, d_2, \ldots, d_k\})$ will also be denoted as $D(d_1, d_2, \ldots, d_k)$ and $G_n(\{d_1, d_2, \ldots, d_k\})$ as $D_n(d_1, d_2, \ldots, d_k)$.

The study of distance graphs was initiated by Eggleton et al. [3]. A large amount of work has focused on colorings of distance graphs [4, 15, 1, 11, 12, 14], but other parameters have also been studied on distance graphs, like the feedback vertex set problem [10].

The aim of this paper is to study the packing chromatic number of infinite distance graphs, with particular emphasis on the case $D = \{1, t\}$. In Section 2, we bound the packing chromatic number of the infinite path power (i.e. infinite distance graph with $D = \{1, 2, \ldots, t\}$). Section 3 concerns packing colorings of distance graphs with $D = \{1, t\}$, for which we prove some lower and upper bounds on the number of colors (see Proposition 1). Exact or sharp results for the packing chromatic number of some other 4-regular distance graphs are presented in Section 4. Section 5 concludes the paper with some remarks and open questions.

Our results about the packing chromatic number of $G(\mathbb{Z}, D)$ for some small values of D (from Sections 2 and 4) are summarized in Table 1.

D	$\chi_\rho \geq$	$\chi_\rho \leq$	period
$1, 2$	8	8	54
$1, 3$	9^*	9	32
$1, 4$	11	16	320
$1, 5$	10^*	12	1028
$1, 6$	12	23	2016
$1, 7$	10^*	15	640
$1, 8$	11^*	25	5184
$1, 9$	10^*	18	576
$1, 2, 3$	17	23	768
$2, 3$	11	13	240
$2, 5$	14	23	336

Table 1: Lower and upper bounds for the packing chromatic number of $G(\mathbb{Z}, D)$ for different values of D. In the fourth column are the periods of the colorings giving the upper bounds. (*: bound obtained by running Algorithm 1 of Section 4).

The bounds of Section 3 are summarized in the following Proposition:

Proposition 1. *Let t, q be integers. Then,*

$$\chi_\rho(D(1,t)) \leq \begin{cases} 89, & t = 2q+1, q \geq 35; \\ 40, & t = 2q+1, q \geq 223; \\ 179, & t = 2q, q \geq 89; \\ 81, & t = 2q, q \geq 224; \\ 29, & t = 96q \pm 1, q \geq 1; \\ 59, & t = 96q+1 \pm 1, q \geq 1. \end{cases}$$

Some proofs of lower bounds use a density argument. For this, we define the density $\rho_a(G_n(D))$ of a color a in $G_n(D)$ as the maximum fraction of vertices colored a in any packing coloring of $G_n(D)$ and $\rho_a(D)$ (or simply ρ_a, if the graph is clear from the context) by $\rho_a(D) = \limsup_{n\to+\infty} \rho_a(G_n(D))$. Let also $\rho_{1,2}(G_n(D))$ be the maximum fraction of vertices colored 1 or 2 in any packing coloring of $G_n(D)$ and let $\rho_{1,2} = \limsup_{n\to+\infty} \rho_{1,2}(G_n(D))$. We have trivially, for any D, $\chi_\rho(G(\mathbb{Z}, D)) \geq \min\{c \mid \sum_{i=1}^{c} \rho_i \geq 1\}$ and $\rho_{1,2} \leq \rho_1 + \rho_2$.

2. Path Powers

The t^{th} power G^t of a graph G is the graph with the same vertex set as G and edges between every vertices x, y that are at a mutual distance of at most t in G. Let $D^t = G(\mathbb{Z}, \{1, 2, \cdots, t\})$ be the t^{th} power of the two-ways infinite path and let $P_n^t = G_n(\{1, 2, \cdots, t\})$ be the t^{th} power of the path P_n on n vertices.

We first present an asymptotic result on the packing chromatic number:

Proposition 2. *$\chi_\rho(D^t) = (1 + o(1))3^t$ and $\chi_\rho(D^t) = \Omega(e^t)$.*

Proof. D^t is a spanning subgraph of the lexicographic product[1] $\mathbb{Z} \circ K_t$ (see Figure 1). Then, as Goddard et al. [9] showed that $\chi_\rho(\mathbb{Z} \circ K_t) = (1 + o(1))3^t$, the same upper bound holds for D^t. To prove the lower bound, since $\rho_i \leq \frac{1}{it+1}$, then for any packing coloring of D^t using at most c colors, c must satisfy:

$$\sum_{i=1}^{c} \frac{1}{it+1} \geq 1.$$

Since

$$\sum_{i=1}^{c} \frac{1}{it+1} < \sum_{i=1}^{c} \frac{1}{it} = \frac{1}{t} \sum_{i=1}^{c} \frac{1}{i} = \frac{H_c}{t},$$

where H_n is the n^{th} harmonic number and since $H_n = \Omega(\ln(n))$, then $\frac{H_c}{t} \geq 1$ implies $c = \Omega(e^t)$. □

Corollary 1. *For any finite subset D of $\mathbb{N}$, the packing chromatic number of $G(\mathbb{Z}, D)$ is finite.*

For very small t, exact values or sharp bounds for the packing chromatic number can be calculated:

[1] the lexicographic product $G \circ H$ of graphs G and H has vertex set $V(G) \times V(H)$ and two vertices (a, x) and (b, y) are linked by an edge if and only if $ab \in E(G)$ or $a = b$ and $xy \in E(H)$

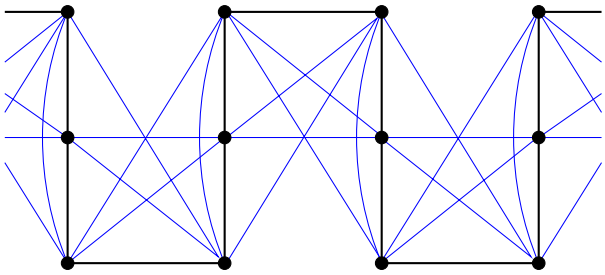

Figure 1: The infinite distance graph D^3 as a subgraph of the lexicographic product $\mathbb{Z} \circ K_3$.

Proposition 3.

$$\chi_\rho(D^2) = 8.$$

Proof. A packing 8-coloring can be constructed by repeating the following pattern of length 54 :

$$8,1,2,6,1,4,3,2,1,5,7,1,2,3,4,1,6,2,1,8,3,1,2,4,1,5,7,$$
$$1,3,2,1,6,4,1,2,3,1,8,5,1,2,4,1,3,6,1,2,7,1,5,4,2,1,3.$$

On the other hand, it can be seen that $\rho_i \leq \frac{1}{2i+1}$ for any $i \geq 1$. However, we next prove that $\rho_{1,2} \leq \frac{1}{2}$. Consider vertices $v, v+1, \ldots, v+9$ for some v. The only possibility to color more than 5 of these 10 vertices is to give color 1 to $v, v+3, v+6, v+9$ and then at most 2 vertices can be given color 2 ($v+1$ or $v+2$, and $v+7$ or $v+8$). But in this case, neither vertex $v+10$ nor vertex $v+11$ can be given color 1 or 2, resulting in 6 vertices colored out of 12. Moreover, an easy computation gives that $\chi_\rho(D^2) \geq \min\{c \mid \frac{1}{2} + \sum_{i=3}^{c} \frac{1}{2i+1} \geq 1\} = 8$. □

Proposition 4.

$$17 \leq \chi_\rho(D^3) \leq 23.$$

Proof. The upper bound comes from a packing 23-coloring of period 768 defined by repeating the sequence of length 768 given in Appendix A.

To prove the lower bound, as the distance $\mathrm{dist}(u,v)$ between the vertices u and v is $\mathrm{dist}(u,v) = \lceil \frac{v-u}{3} \rceil$, then $\rho_i \leq \frac{1}{3i+1}$ and an easy computation gives that $\chi_\rho(D^3) \geq \min\{c| \sum_{i=1}^{c} \frac{1}{3i+1} \geq 1\} = 17$. □

3. $D(1,t)$ with large t

The general method is to cut the distance graph into sets of consecutive vertices of size $s = t-1$ or $s = t+1$, depending on the value of t and to color each set by a predefined color pattern. Let s be either $t+1$ or $t-1$ and let $A_i = \{is, is+1, \ldots, (i+1)s-1\}$ and B_i be the subgraph of $D(1,t)$ induced by A_i. Notice that $V(D(1,t)) = \bigcup_{i=-\infty}^{+\infty} A_i$ and that if $s = t+1$, then each B_i is an induced cycle of $D(1,t)$ of length $s = t+1$ (see Figure 2). By a *color pattern* P, we mean a sequence of integers $(c_1, c_2, \ldots, c_s)$ of length s that will be associated to some subgraph B_i by giving the color c_j to the j^{th} vertex of B_i. If S is a sequence of integers, S^p is the sequence obtained by repeating S p times. The *cyclic distance* between elements s_i and s_j of a sequence $(s_1, s_2, \ldots, s_\ell)$ is $\min(|j-i|, \ell - |j-i|)$.

We first need to know the distance between two vertices in $D(1,t)$.

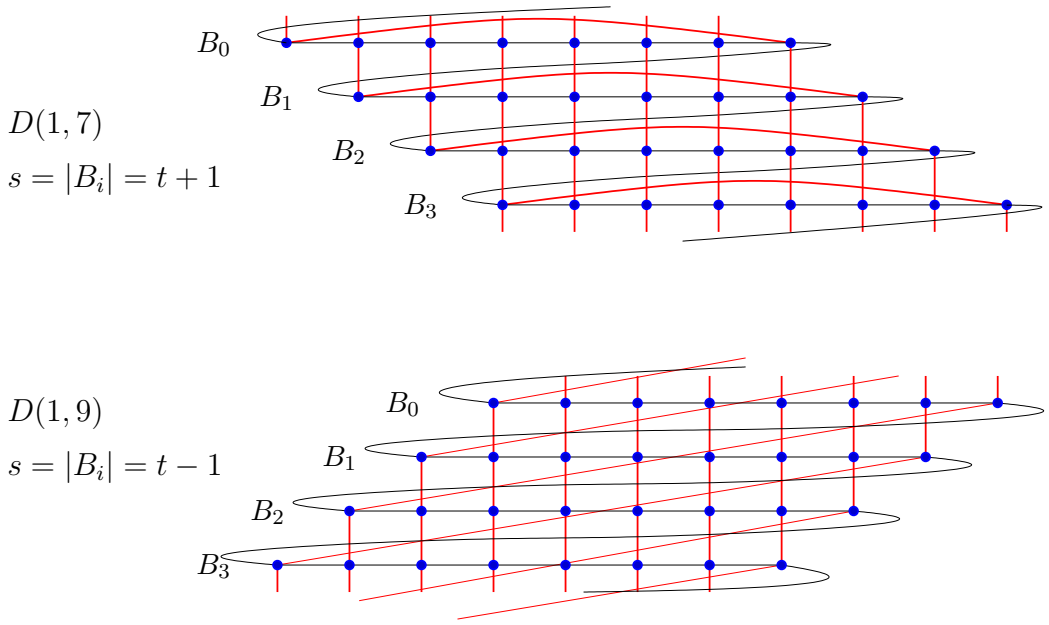

Figure 2: $D(1,t)$, with $t = 7$ (on the top) and $t = 9$ (on the bottom) drawn by rows of size $s = 8$.

Lemma 1. *The distance between two vertices u and v of $D(1,t)$ is $dist(u,v) = \min(q+r, q+1+t-r)$, where $|v-u| = qt+r$, with $0 \leq r < t$.*

Proof. Let us call an edge joining vertices x and y, with $|y-x| = k$ a k-edge. Assume, without loss of generality, that $v \geq u$. then, any minimal path between u and v uses either q t-edges and r 1-edges or $q+1$ t-edges and $t-r$ 1-edges. □

The key lemma of our method is the following one which gives conditions for a coloring of $D(1,t)$ by color patterns to be a packing coloring.

Lemma 2. *Let $s > 1$ be a positive integer and for each integer i, set $A_i = \{is, is+1, \ldots, (i+1)s-1\}$. Let t be a positive integer and for each i, let B_i be the subgraph of $G = D(1,t)$ induced by A_i, and C_i be the graph B_i with an additional edge joining vertices is and $(i+1)s-1$ if $s = t-1$. Suppose that G is colored in such a way that:*

i) *for each integer i, the coloring inherited by each C_i is a packing coloring;*

ii) *for each pair of integers i and j, if c is the maximum common color in both C_i and C_j then we have $c < s$, $|i-j| > \frac{c}{2}$, and each $b \leq c$ that is a common color in both C_i and C_j has the property that $si+k$ is colored b if and only if $sj+k$ is colored b for each $k \in \{0, 1, \ldots, s-1\}$.*

Then the coloring is a packing coloring of G whenever t is in $\{s+1, s-1\}$.

Proof. Suppose vertices u and v have the same color, say e, and, without loss of generality, assume u is in B_0. Let $\sigma : V(G) \to V(C_0)$ be defined by $\sigma(k) = k \bmod s$ for each $k \in \mathbb{N}$. Observe that when $t = s + 1$ or $t = s - 1$, if two vertices x and y are adjacent in G, then $\sigma(x)$ and $\sigma(y)$ are adjacent in C_0. But then a path in G between u and v maps via σ to a path of at most the same length between two vertices in C_0 colored e. Since, by hypothesis, C_0 is colored by a packing coloring, as long as $u \neq \sigma(v)$, the distance between u and v must be greater than e.

If $u = \sigma(u) = \sigma(v)$, then $v - u = js$ for some j. If $s = t - 1$, then $v - u = j(t-1) = (j-1)t + t - j$ and by Lemma 1, $\text{dist}(u, v) = \min(j - 1 + t - j, j + t - t + j) = \min(t - 1, 2j) > e$ since by hypothesis, $e < s = t - 1$ and $2j > e$. Else, if $s = t + 1$ then $v - u = j(t+1) = jt + j$ and by Lemma 1, $\text{dist}(u, v) = \min(j + j, j + 1 + t - j) = \min(2j, t + 1) > e$ by hypothesis. □

3.1. Proof of Proposition 1

Proof. Let t be an integer, $G = D(1, t)$ and $s = 4p$ if $t = 4p - 1$ or $t = 4p + 1$ for some p; $s = 4p + 1$ if $t = 4p$ or $t = 4p + 2$. For each integer i, set $A_i = \{is, is + 1, \ldots, (i + 1)s - 1\}$ and let B_i be the subgraph of G induced by A_i.

In each of the following cases, a packing coloring of G is defined by assigning to each subgraph B_i a pattern of colors with length s. We will use the following sub-patterns of colors:

$S_{2,3} = (1, 2, 1, 3)$,
$S_{4,9} = (1, 4, 1, 5, 1, 8, 1, 4, 1, 5, 1, 9)$,
$S_{4,11} = (1, 4, 1, 5, 1, 10, 1, 4, 1, 5, 1, 11)$,
$S_{6,15} = (1, 6, 1, 7, 1, 12, 1, 13, 1, 6, 1, 7, 1, 14, 1, 15)$,
$S_{6,21} = (1, 6, 1, 7, 1, 16, 1, 17, 1, 6, 1, 7, 1, 18, 1, 19, 1, 6, 1, 7, 1, 20, 1, 21)$,
$S_{6,29} = (1, 6, 1, 7, 1, 22, 1, 23, 1, 6, 1, 7, 1, 24, 1, 25, 1, 6, 1, 7, 1, 26, 1, 27, 1, 6, 1, 7, 1, 28, 1, 29)$,
$S_{6,31} = (1, 6, 1, 7, 1, 22, 1, 23, 1, 6, 1, 7, 1, 24, 1, 25, 1, 6, 1, 7, 1, 26, 1, 27, 1, 6, 1, 7, 1, 22, 1, 23,$
$1, 6, 1, 7, 1, 28, 1, 29, 1, 6, 1, 7, 1, 30, 1, 31)$.

Case A. t **is odd.** First, since $s = 4p$ for some integer p and thanks to Lemma 2, we can assign to each subgraph B_{2i+1} the color pattern $(S_{2,3})^p$. In order to color subgraphs B_{2i}, we consider three sub-cases (that are not totally disjoints).

Subcase A.1. $t = 96q \pm 1$ **for some** $q \geq 1$**.** A packing coloring of $D(1, t)$ using these sub-patterns is constructed by assigning inductively to 8 consecutive subgraphs B_{2i} the sequence of color patterns

$$\mathcal{P} = ((S_{4,9})^{8q}, (S_{6,15})^{6q}, (S_{4,11})^{8q}, (S_{6,21})^{4q}, (S_{4,9})^{8q}, (S_{6,15})^{6q}, (S_{4,11})^{8q}, (S_{6,29})^{3q}).$$

Since the cyclic distance between two occurrences of any color e in each color pattern is always greater than e, then Condition i) of Lemma 2 is satisfied. Moreover, as the cyclic distance between any two color patterns in $\mathcal{P}$ is always greater than a quarter (since color patterns of $\mathcal{P}$ are associated only with subgraphs of even indices) of their maximum common color, then Condition ii) is also satisfied. Hence, the coloring is a packing coloring of $D(1, t)$ and $\chi_\rho(D(1, t)) \leq 29$.

Subcase A.2. $t = 2p+1$ **for some** $p \geq 223$**.** We denote by $S \sqcup (1,\alpha)^r$ any sequence obtained by inserting r quasi evenly cyclically-distributed occurrences of the pair $(1,\alpha)$ in the sequence S; insertions being made only after a color different from 1, in order to keep the sequence alternate between color 1 and other colors.
For example, $(1,4,1,5,1,8,1,4,1,5,1,9)^3 \sqcup (1,\alpha)^5$ can be rewritten as
$(1,4,1,5,1,8,\mathbf{1,\alpha,}1,4,1,5,1,9,\mathbf{1,\alpha,}1,4,1,5,1,8,1,4,\mathbf{1,\alpha,}1,5,1,9,1,4,1,5,\mathbf{1,\alpha,}1,8,1,4,1,5,1,9,\mathbf{1,\alpha}).$

Then, color patterns using colors from $\{1,2,\ldots,40\}$ are defined by:
$Q_1^i = (S_{4,9})^{q_1} \sqcup (1,32+i)^{r_1}$, for $s = 12q_1 + 2r_1, 0 \leq r_1 \leq 4$, $i = 0,1,2$;
$Q_2^i = (S_{4,11})^{q_2} \sqcup (1,35+i)^{r_2}$, for $s = 12q_2 + 2r_2, 0 \leq r_2 \leq 4$, $i = 0,1,2$;
$Q_3^i = (S_{6,15})^{q_3} \sqcup (1,38+i)^{r_3}$, for $s = 16q_3 + 2r_3, 0 \leq r_3 \leq 6$, $i = 0,1,2$;
$Q_4 = (S_{6,21})^{q_4} \sqcup (1,30)^{r_4}$, for $s = 24q_4 + 2r_4, 0 \leq r_4 \leq 10$;
$Q_5 = (S_{6,29})^{q_5} \sqcup (1,31)^{r_5}$, for $s = 32q_5 + 2r_5, 0 \leq r_5 \leq 14$;

and we assign inductively to 24 consecutive subgraphs B_{2i} the sequence of color patterns $\mathcal{Q}$ defined by

$$\mathcal{Q} = (Q_1^0, Q_3^0, Q_2^0, Q_4, Q_1^1, Q_3^1, Q_2^1, Q_5, Q_1^2, Q_3^2, Q_2^2, Q_4, Q_1^0, Q_3^0, Q_2^0, Q_5, Q_1^1, Q_3^1, Q_2^1, Q_4, Q_1^2, Q_3^2, Q_2^2, Q_5).$$

In order for a color pattern $S \sqcup (1,\alpha)^r$ to satisfy Condition i) of Lemma 2 and as the pairs $(1,\alpha)$ have to be inserted only on even positions, we must have $2\lfloor\lfloor\frac{|S|}{r}\rfloor/2\rfloor \geq \alpha$. Hence the worst case for this separation constraint is for color 31 in Q_5 when $r_5 = 14$: one can insert 14 occurrences of $(1,31)$ if $2\lfloor\lfloor\frac{32q_5}{14}\rfloor/2\rfloor \geq 31$, which is true as soon as $q_5 = 14$ and thus $s = 448$. Moreover, it can be seen that the added color in each pattern is chosen in such a way that Condition ii) is satisfied. Hence, the coloring is a packing coloring of $D(1,t)$ and $\chi_\rho(D(1,t)) \leq 40$.

Subcase A.3. $t = 2p+1$ **for some** p**,** $35 \leq p \leq 222$**.** The base case is $s \equiv 0 \pmod{48}$ for which the sequence of color patterns that is assigned inductively to 8 consecutive subgraphs B_{2i} is defined as follows:

$$\mathcal{R} = (R_1, R_3, R_2, R_4, R_1, R_3, R_2, R_5),$$

with $R_1 = (S_{4,9})^{4q}$, $R_2 = (S_{4,11})^{4q}$, $R_3 = (S_{6,15})^{3q}$, $R_4 = (S_{6,21})^{2q}$, and $R_5 = (S_{6,31})^{q}$.

As for Subcase A.1, it can be easily checked that the defined coloring is a packing coloring.

Now, for $s \not\equiv 0 \pmod{48}$, we may replace each of the above color patterns $R_j \in \mathcal{R}$ by a certain number of patterns R_j^i (depending on the residue of s modulo the length of the sub-pattern used) that will be used in turn, as for Subcase A.2.

Let ϵ be the empty sequence and let c_j and δ_j, $1 \leq j \leq 5$ be some integers (that will be set just after).

Set $R_1^i = (S_{4,9})^{q_1}.T_1^i$, with $s = 12q_1 + 4r_1$, $0 \leq r_1 < 3$, $0 \leq i < \delta_1$, and

$$T_1^i = \begin{cases} \epsilon, & \text{if } r_1 = 0; \\ (1, c_1+i, 1, c_1+\delta_1+i), & \text{if } r_1 = 1; \\ (1,4,1,5,1,c_1+i,1,c_1+\delta_1+i), & \text{if } r_1 = 2. \end{cases}$$

Set $R_2^i = (S_{4,11})^{q_2}.T_2^i$, with $s = 12q_2 + 4r_2$, $0 \leq r_2 < 3$, $0 \leq i < \delta_2$, and

$$T_2^i = \begin{cases} \epsilon, & \text{if } r_2 = 0; \\ (1, c_2+i, 1, c_2+\delta_2+i), & \text{if } r_2 = 1; \\ (1,4,1,5,1,c_2+i,1,c_2+\delta_2+i), & \text{if } r_2 = 2. \end{cases}$$

Set $R_3^i = (S_{6,15})^{q_3}.T_3^i$, with $s = 16q_3 + 4r_3$, $0 \le r_3 < 4$, $0 \le i < \delta_3$, and

$$T_3^i = \begin{cases} \epsilon, & \text{if } r_3 = 0; \\ (1, c_3 + i, 1, c_3 + \delta_3 + i), & \text{if } r_3 = 1; \\ (1, 6, 1, 7, 1, c_3 + i, 1, c_3 + \delta_3 + i), & \text{if } r_3 = 2; \\ (1, 6, 1, 7, 1, c_3 + i, 1, c_3 + \delta_3 + i, 1, c_3 + 2\delta_3 + i, 1, c_3 + 3\delta_3 + i), & \text{if } r_3 = 3. \end{cases}$$

Set $R_4^i = (S_{6,21})^{q_4}.T_4^i$, with $s = 24q_4 + 4r_4$, $0 \le r_4 < 6$, $0 \le i < \delta_4$, and

$$T_4^i = \begin{cases} \epsilon, & \text{if } r_4 = 0; \\ (1, c_4 + i, 1, c_4 + \delta_4 + i), & \text{if } r_4 = 1; \\ (1, 6, 1, 7, 1, c_4 + i, 1, c_4 + \delta_4 + i), & \text{if } r_4 = 2; \\ (1, 6, 1, 7, 1, c_4 + i, 1, c_4 + \delta_4 + i, 1, c_4 + 2\delta_4 + i, 1, c_4 + 3\delta_4 + i), & \text{if } r_4 = 3; \\ (1, 6, 1, 7, 1, c_4 + i, 1, c_4 + \delta_4 + i, 1, 6, 1, 7, 1, c_4 + 2\delta_4 + i, 1, c_4 + 3\delta_4 + i), & \text{if } r_4 = 4; \\ (1, 6, 1, 7, 1, c_4 + i, 1, c_4 + \delta_4 + i, 1, 6, 1, 7, 1, c_4 + 2\delta_4 + i, 1, c_4 + 3\delta_4 + i, & \\ \quad 1, c_4 + 4\delta_4 + i, 1, c_4 + 5\delta_4 + i), & \text{if } r_4 = 5; \end{cases}$$

Set $R_5^i = (S_{6,31})^{q_5 - 1}.T_5^i$, with $s = 48q_5 + 4r_5$, $0 \le r_5 < 12$, $0 \le i < \delta_5$, and

$$T_5^i = \begin{cases} S_{6,31}, & \text{if } r_5 = 0; \\ S_{6,31}.(1, c_5 + i, 1, c_5 + \delta_5 + i), & \text{if } r_5 = 1; \\ S_{6,31}.(1, 6, 1, 7, 1, c_5 + i, 1, c_5 + \delta_5 + i), & \text{if } r_5 = 2; \\ S_{6,31}.(1, 6, 1, 7, 1, c_5 + i, 1, c_5 + \delta_5 + i, 1, c_5 + 2\delta_5 + i, 1, c_5 + 3\delta_5 + i), & \text{if } r_5 = 3; \\ (S_{6,29})^2, & \text{if } r_5 = 4; \\ (S_{6,29})^2.(1, c_5 + i, 1, c_5 + \delta_5 + i), & \text{if } r_5 = 5; \\ (S_{6,29})^2.(1, 6, 1, 7, 1, c_5 + i, 1, c_5 + \delta_5 + i), & \text{if } r_5 = 6; \\ (S_{6,29})^2.(1, 6, 1, 7, 1, c_5 + i, 1, c_5 + \delta_5 + i, 1, c_5 + 2\delta_5 + i, 1, c_5 + 3\delta_5 + i), & \text{if } r_5 = 7; \\ S_{6,31}.S_{6,29}, & \text{if } r_5 = 8; \\ S_{6,31}.S_{6,29}.(1, c_5 + i, 1, c_5 + \delta_5 + i), & \text{if } r_5 = 9; \\ S_{6,31}.S_{6,29}.(1, 6, 1, 7, 1, c_5 + i, 1, c_5 + \delta_5 + i), & \text{if } r_5 = 10; \\ S_{6,31}.S_{6,29}.(1, 6, 1, 7, 1, c_5 + i, 1, c_5 + \delta_5 + i, 1, c_5 + 2\delta_5 + i, 1, c_5 + 3\delta_5 + i), & \text{if } r_5 = 11; \end{cases}$$

As the cyclic distance between two occurrences of either the color pattern R_1 or of R_2 or of R_3 in $\mathcal{R}$ is equal to 4 (hence, each of these three patterns appears every 8 set B_i), and if e is the maximum color used in R_j^i, then, according to Lemma 2, for $j = 1, 2, 3$, δ_j must satisfy

$$\delta_j \ge \begin{cases} 1, \text{ if } e \le 15; \\ 2, \text{ if } 16 \le e \le 31; \\ 3, \text{ if } 32 \le e \le 47; \\ 4, \text{ if } 48 \le e \le 63; \\ 5, \text{ if } 64 \le e \le 79. \end{cases}$$

Similarly, the cyclic distance between two occurrences of either the color pattern R_4 or of R_5 in $\mathcal{R}$ is equal to 8, hence, for $j = 4$ or 5, δ_j must satisfy

$$\delta_j \ge \begin{cases} 1, \text{ if } e \le 31; \\ 2, \text{ if } 32 \le e \le 63; \\ 3, \text{ if } 64 \le e \le 95. \end{cases}$$

Therefore, for each residue of s modulo 48, a packing coloring is obtained by fixing the values of c_j and δ_j as indicated in the next table (δ_j is set to the smallest value satisfying the above inequations). The largest color used in each case is reported on the last row.

s (mod 48)	0	4	8	12	16	20	24	28	32	36	40	44
c_1, δ_1	/	32, 3	32, 3	/	32, 3	32, 3	/	32, 3	32, 3	/	32, 3	32, 3
c_2, δ_2	/	38, 3	38, 3	/	38, 3	38, 3	/	38, 3	38, 3	/	38, 3	38, 3
c_3, δ_3	/	44, 4	44, 4	32, 3	/	44, 4	32, 3	44, 4	/	32, 3	44, 4	44, 4
c_4, δ_4	/	52, 2	52, 2	44, 2	44, 2	52, 2	/	60, 2	44, 2	38, 2	52, 2	60, 2
c_5, δ_5	/	56, 2	56, 2	52, 2	/	64, 3	38, 2	64, 3	/	46, 2	60, 2	78, 2
largest color	31	59	59	59	51	69	41	75	47	49	63	89

An illustration for the case $s \equiv 28 \pmod{48}$ is given in Appendix B.

Case B. t is even. For $t = 4p$ or $t = 4p+2$, recall that subgraphs B_i are of size $s = 4p+1$. New color patterns are constructed by inserting a new color at the end of each pattern (of length $s' = s - 1 = 4p$) defined in Subcases A.1, A.2 and A.3.

By Lemma 2, the problem of adding the missing color in each color pattern defined in subcases A.1, A.2 and A.3 is equivalent to the one of coloring the infinite path P_∞ with colors from $\{k_1, k_1+1, \ldots, k_2\}$ such that vertices of color e are at distance greater than $\frac{e}{2}$.

We are going to show, by induction on k_1, that $k_2 \leq 2k_1 - 1$. For $k_1 = 2$, vertices can be colored by alternating color 2 and color 3, so $k_2 = 3$. Assume that P_∞ can be colored with colors from $\{k_1, k_1+1, \ldots, k_2 \leq 2k_1 - 1\}$ and let $k_1' = k_1 + 1$. Replace now color k_1 by colors k_2+1 and k_2+2 alternatively. Then the largest color used is $k_2' = k_2 + 2 \leq 2k_1 + 1 = 2k_1' - 1$ and the constraint is satisfied since if vertices x and y are colored $k_2 + 2$ then their mutual distance satisfies $\text{dist}(x, y) > 2\frac{k_1}{2} \geq \frac{k_2+1}{2} > \frac{k_2}{2}$.

As the colorings defined in Subcase A.1 (Subcases A.2 and A.3, respectively) use colors from 1 to 29 (40 and at most 89, respectively), then we obtain a packing coloring of $D(1, t)$ with colors from 1 to at most $2 \times 30 - 1 = 59$ (81 and 179, respectively), provided that $t \geq 96$ (448 and 144, respectively).

□

Remark 1.

- *In Subcase A.2, the method can produce a packing coloring using less than* 40 *colors, depending on the value of s (i.e. if some r_i are equal to zero).*
- *A combination of the methods of Subcases A.2 and A.3 could be used to define a packing coloring for odd t, $95 \leq t \leq 447$, using less colors than in Subcase A.3.*
- *For Case B, it seems that less than $2k_1 - 1$ colors are sufficient for such a coloring. When $k_1 = 90$, a computation gives $k_2 = 156$ for such a coloring; when $k_1 = 41$, we find $k_2 = 72$ and when $k_1 = 30$, we find $k_2 = 53$.*

4. $D(a, b)$ with small a and b

The results from Section 3 do not apply for $D(1, t)$ with small t, however it is possible to derive exact or sharp results for some of them, using density arguments and the computer.

Algorithm 1 is a simple algorithm that prints all the packing k-colorings of $D_n(1, t)$. It checks, for each vertex, each possible color in a recursive fashion. Hence it must be used by initializing the first n elements of the array `color` to 0 and calling RecColor(0).

Proposition 5.

$$\chi_\rho(D(1, 3)) = 9.$$

Algorithm 1: RecColor(i)

```
Data: global integers n, k, t; global array color;
if i = n then
    print(color);
else
    for c from 1 to k do
        if ∄ j < i such that color[j]=c and dist(i, j) ≤ c then
            color[i] ← c;
            RecColor(i + 1);
            color[i] ← 0;
```

Proof. first, remark that the graph-distance $\text{dist}(i,j)$ between vertex i and vertex $j \geq i$ is $\text{dist}(i,j) = \lfloor \frac{j-i}{3} \rfloor + (j-i) \pmod 3$.

A packing 9-coloring of $D(1,3)$ of period 32 is given by the following sequence:

$$1,2,1,3,1,4,1,5,1,2,1,3,1,6,1,7,1,2,1,3,1,4,1,5,1,2,1,3,1,8,1,9.$$

It is routine to check that the vertices of a same color are sufficiently distant. On the other hand, running an implementation of Algorithm 1 with $n = 100$, $k = 8$, and $t = 3$, outputs no coloring, showing that 8 colors are not sufficient for a packing coloring of $D_{100}(1,3)$.

□

Proposition 6.

$$11 \leq \chi_\rho(D(1,4)) \leq 16;$$
$$10 \leq \chi_\rho(D(1,5)) \leq 12;$$
$$12 \leq \chi_\rho(D(1,6)) \leq 23;$$
$$10 \leq \chi_\rho(D(1,7)) \leq 15;$$
$$11 \leq \chi_\rho(D(1,8)) \leq 25;$$
$$10 \leq \chi_\rho(D(1,9)) \leq 18.$$

Proof. For the upper bounds, packing k-colorings are defined by exhibiting a pattern using colors from $\{1,\cdots,k\}$ of length ℓ for each case. For $D(1,4)$, the pattern with $k = 16$ and $\ell = 320$ is given in Appendix A. For $D(1,5)$ ($D(1,6)$, $D(1,7)$, $D(1,8)$, and $D(1,9)$, respectively), the pattern with $(k,\ell) = (12,1028)$ ($(23,2016)$, $(15,640)$, $(25,5184)$, $(18,576)$, respectively) can be found at `http://www.u-bourgogne.fr/o.togni/PCDG.html`.

For the lower bounds, we use either density arguments or computer running Algorithm 1.

For $D(1,4)$, we have $\rho_1 \leq \frac{2}{5}$ since at most 2 out of 5 consecutive vertices can be colored 1. Moreover, $\rho_i \leq \frac{1}{4i-2}$ for $i \geq 2$ and $\min\{c|\ \frac{2}{5} + \sum_{i=2}^{c} \frac{1}{4i-2} \geq 1\} = 11$.

For $D(1,5)$, running an implementation of Algorithm 1 with $n = 43$, $k = 9$, and $t = 5$ outputs no coloring. Hence $\chi_\rho(D(1,5)) \geq 10$.

For $D(1,6)$, we have $\rho_1 \leq \frac{3}{7}$ since at most 3 out of 7 consecutive vertices can be colored 1. We now show that $\rho_2 \leq \frac{2}{11}$. Let v be a vertex colored 2. If $v+3$ is also colored 2, then no vertex of $\{v+4,\cdots,v+10\}$ can be colored 2. Hence 2 vertices out of 11 are colored 2.

If $v+3$ is not colored 2 but $v+4$ is, then only one of $v+8$, $v+14$ can be colored 2 among $\{v+5,\cdots,v+16\}$, resulting in 3 out of 17 vertices colored 2 and $\frac{3}{17}<\frac{2}{11}$. If neither $v+3$ nor $v+4$ is colored 2 then no vertex of $\{v+5,v+6,v+7\}$ can be colored 2 and at most one vertex of $\{v+8,v+9,v+10\}$ can have color 2, resulting in 2 vertices out of 11 colored 2. Moreover, if $i\geq 3$, then $\rho_i\leq\frac{1}{6i-9}$ and $\min\{c|\ \frac{3}{7}+\frac{2}{11}+\sum_{i=3}^{c}\frac{1}{6i-9}\geq 1\}=12$.

For $D(1,7)$, running an implementation of Algorithm 1 with $n=44$, $k=9$, and $t=7$ outputs no coloring. Hence $\chi_\rho(D(1,7))\geq 10$.

For $D(1,8)$, running an implementation of Algorithm 1 with $n=41$, $k=10$, and $t=8$ outputs no coloring. Hence $\chi_\rho(D(1,8))\geq 11$.

For $D(1,9)$, running an implementation of Algorithm 1 with $n=46$, $k=9$, and $t=9$ outputs no coloring. Hence $\chi_\rho(D(1,9))\geq 10$. □

It is interesting to notice that sometimes adding just one more color allows us to shorten considerably the period of the packing coloring, as can be seen with $D(1,5)$ with the following periodic packing 13-coloring of period 80 (compared with the packing 12-coloring of period 1028):

$1,2,1,3,1,4,1,5,1,2,1,3,1,6,1,7,1,2,1,3,1,10,1,4,1,2,1,3,1,5,1,11,1,2,1,3,1,8,1,9,$
$1,2,1,3,1,4,1,5,1,2,1,3,1,6,1,7,1,2,1,3,1,12,1,4,1,2,1,3,1,5,1,13,1,2,1,3,1,9,1,8.$

We now turn our attention to other 4-regular distance graphs, i.e. graphs of type $D(a,b)$, with $2\leq a\leq b$. First, remark that if a and b are not co-prime, then the graph $D(a,b)$ is not connected and consists in $g=\gcd(a,b)$ copies of $D(\frac{a}{g},\frac{b}{g})$. Hence we only consider distance graphs $D(a,b)$ with $\gcd(a,b)=1$.

The smallest example is $D(2,3)$ which is a subgraph of $D(1,2,3)=P_\infty^3$, thus $\chi_\rho(D(2,3))\leq\chi_\rho(P_\infty^3)\leq 23$. In fact, we show that the upper bound is much less than 23:

Proposition 7.

$$11\leq\chi_\rho(D(2,3))\leq 13;$$
$$14\leq\chi_\rho(D(2,5))\leq 23.$$

Proof. The lower bound $11\leq\chi_\rho(D(2,3))$ is obtained by calculating the maximum density ρ_i of a color i: it can be seen that $\rho_1=\frac{2}{5}$ and $\rho_i=\frac{1}{3i+1}$ for $i\geq 2$ and that $\min\{c|\ \frac{2}{5}+\sum_{i=2}^{c}\frac{1}{3i+1}\geq 1\}=11$.

For the lower bound $14\leq\chi_\rho(D(2,5))$, it can be seen that $\rho_1=\frac{3}{7}$ and $\rho_i=\frac{1}{5i-4}$ for $i\geq 2$ and that $\min\{c|\ \frac{2}{5}+\sum_{i=2}^{c}\frac{1}{5i-4}\geq 1\}=14$.

The upper bounds come from the packing 13-coloring of $D(2,3)$ of period 240 and the packing 23-coloring of $D(2,5)$ of period 336 given in Appendix A. □

5. Concluding remarks

We have shown that the packing chromatic number of any infinite distance graph with finite D is finite and is at most 40 (81, respectively) for $D=\{1,t\}$ with t being an odd (even, respectively) integer greater than or equal to 447.

Among the many possible research directions, one can try to find better bounds and/or more simple methods for $D(1,t)$. In fact, running a simple greedy packing coloring algorithm that consists in coloring vertices of a distance graph one-by-one from the left to the right

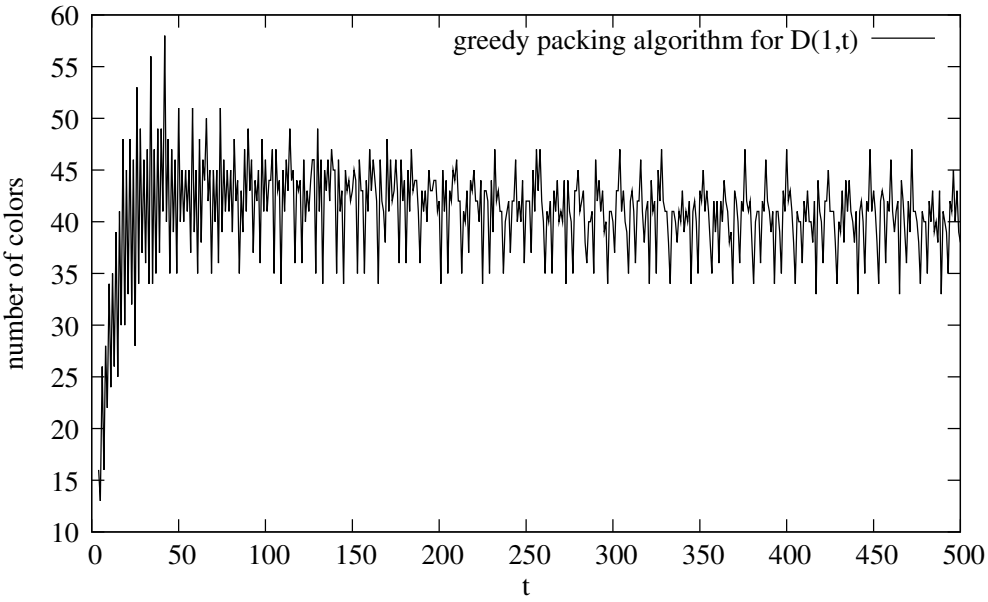

Figure 3: Number of colors for a packing coloring of $D(1,t)$ using a greedy algorithm.

with the smallest color with respect to the constraint, suggests that the upper bounds found in Section 3 can be strengthened. Figure 3 shows the number of colors used by the greedy algorithm for a packing coloring of $D_n(1,t)$ (with $n = 1000000$) as a function of t for the first 500 values of t. One can see on the figure that for large t, the algorithm finds a packing coloring, using between 30 and 50 colors. Moreover, more colors are needed in general when t is even compared to when t is odd. But surprisingly, even if we look only at even (or odd) values of t, the function is not monotonic. We wonder if the same goes for χ_ρ. An interesting future work would be to study in more details the behavior of this greedy algorithm.

Finally, a summary of the values of t for which a upper bound on the the packing chromatic number of $D(1,t)$ is known and those that remain to be found is presented in Table 2.

odd t	$11 \to 45$	$47, 49$	$51 \to 69$	$71 \to 445$	$447 \to +\infty$
$\chi_\rho \leq$	?	31	?	between 29 and 89	40
even t	$10 \to 94$	$96, 98$	$100 \to 142$	$144 \to 446$	$448 \to +\infty$
$\chi_\rho \leq$	?	59	?	between 59 and 179	81

Table 2: Known upper bounds for the packing chromatic number of $D(1,t)$

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A. Periodic packing coloring of some distance graphs

A periodic packing 23-coloring of $D(1,2,3)$ of period 768

23, 1, 4, 5, 3, 1, 2, 6, 7, 1, 9, 10, 12, 1, 2, 3, 4, 1, 8, 5, 13, 1, 2, 14, 16, 1, 3, 6, 11, 1, 2, 4, 7, 1, 15, 5, 3, 1, 2, 9, 18, 1, 10, 8, 4, 1, 2, 3, 6, 1, 12, 5, 17, 1, 2, 7, 19, 1, 3, 4, 13, 1, 2, 11, 20, 1, 14, 5, 3, 1, 2, 6, 4, 1, 8, 9, 10, 1, 2, 3, 7, 1, 15, 5, 16, 1, 2, 4, 12, 1, 3, 6, 21, 1, 2, 18, 22, 1, 11, 5, 3, 1, 2, 4, 7, 1, 8, 9, 10, 1, 2, 3, 6, 1, 13, 5, 4, 1, 2, 14, 17, 1, 3, 19, 23, 1, 2, 7, 12, 1, 4, 5, 3, 1, 2, 6, 8, 1, 9, 10, 11, 1, 2, 3, 4, 1, 15, 5, 16, 1, 2, 7, 18, 1, 3, 6, 13, 1, 2, 4, 20, 1, 8, 5, 3, 1, 2, 9, 12, 1, 10, 14, 4, 1, 2, 3, 6, 1, 7, 5, 11, 1, 2, 17, 19, 1, 3, 4, 8, 1, 2, 21, 15, 1, 22, 5, 3, 1, 2, 6, 4, 1, 7, 9, 10, 1, 2, 3, 12, 1, 13, 5, 16, 1, 2, 4, 8, 1, 3, 6, 11, 1, 2, 14, 7, 1, 18, 5, 3, 1, 2, 4, 9, 1, 20, 10, 17, 1, 2, 3, 6, 1, 8, 5, 4, 1, 2, 7, 12, 1, 3, 13, 15, 1, 2, 11, 19, 1, 4, 5, 3, 1, 2, 6, 9, 1, 10, 8, 14, 1, 2, 3, 4, 1, 7, 5, 16, 1, 2, 21, 22, 1, 3, 6, 18, 1, 2, 4, 12, 1, 11, 5, 3, 1, 2, 8, 7, 1, 9, 10, 4, 1, 2, 3, 6, 1, 13, 5, 15, 1, 2, 14, 17, 1, 3, 4, 19, 1, 2, 7, 8, 1, 20, 5, 3, 1, 2, 6, 4, 1, 9, 10, 11, 1, 2, 3, 12, 1, 16, 5, 18, 1, 2, 4, 7, 1, 3, 6, 8, 1, 2, 13, 21, 1, 14, 5, 3, 1, 2, 4, 9, 1, 10, 15, 17, 1, 2, 3, 6, 1, 7, 5, 4, 1, 2, 8, 11, 1, 3, 12, 19, 1, 2, 20, 22, 1, 4, 5, 3, 1, 2, 6, 7, 1, 9, 10, 13, 1, 2, 3, 4, 1, 8, 5, 14, 1, 2, 16, 18, 1, 3, 6, 11, 1, 2, 4, 7, 1, 12, 5, 3, 1, 2, 9, 15, 1, 10, 8, 4, 1, 2, 3, 6, 1, 17, 5, 13, 1, 2, 7, 19, 1, 3, 4, 20, 1, 2, 11, 14, 1, 21, 5, 3, 1, 2, 6, 4, 1, 8, 9, 10, 1, 2, 3, 7, 1, 12, 5, 16, 1, 2, 4, 15, 1, 3, 6, 13, 1, 2, 18, 22, 1, 11, 5, 3, 1, 2, 4, 7, 1, 8, 9, 10, 1, 2, 3, 6, 1, 14, 5, 4, 1, 2, 12, 17, 1, 3, 19, 20, 1, 2, 7, 23, 1, 4, 5, 3, 1, 2, 6, 8, 1, 9, 10, 11, 1, 2, 3, 4, 1, 13, 5, 15, 1, 2, 7, 16, 1, 3, 6, 12, 1, 2, 4, 14, 1, 8, 5, 3, 1, 2, 9, 18, 1, 10, 21, 4, 1, 2, 3, 6, 1, 7, 5, 11, 1, 2, 17, 19, 1, 3, 4, 8, 1, 2, 13, 20, 1, 12, 5, 3, 1, 2, 6, 4, 1, 7, 9, 10, 1, 2, 3, 14, 1, 15, 5, 16, 1, 2, 4, 8, 1, 3, 6, 11, 1, 2, 18, 7, 1, 22, 5, 3, 1, 2, 4, 9, 1, 12, 10, 13, 1, 2, 3, 6, 1, 8, 5, 4, 1, 2, 7, 17, 1, 3, 14, 19, 1, 2, 11, 15, 1, 4, 5, 3, 1, 2, 6, 9, 1, 10, 8, 16, 1, 2, 3, 4, 1, 7, 5, 12, 1, 2, 13, 18, 1, 3, 6, 20, 1, 2, 4, 21, 1, 11, 5, 3, 1, 2, 8, 7, 1, 9, 10, 4, 1, 2, 3, 6, 1, 14, 5, 15, 1, 2, 17, 19, 1, 3, 4, 12, 1, 2, 7, 8, 1, 13, 5, 3, 1, 2, 6, 4, 1, 9, 10, 11, 1, 2, 3, 16, 1, 18, 5, 22, 1, 2, 4, 7, 1, 3, 6, 8, 1, 2, 14, 20, 1, 12, 5, 3, 1, 2, 4, 9, 1, 10, 13, 15, 1, 2, 3, 6, 1, 7, 5, 4, 1, 2, 8, 11, 1, 3, 17, 19, 1, 2, 21

A periodic packing 16-coloring of $D(1,4)$ of period 320

1, 2, 1, 3, 4, 1, 5, 1, 2, 7, 1, 6, 1, 3, 2, 1, 8, 1, 4, 10, 1, 2, 1, 3, 5, 1, 9, 1, 2, 12, 1, 13, 1, 3, 2, 1, 4, 1, 6, 7, 1, 2, 1, 3, 11, 1, 5, 1, 2, 8, 1, 4, 1, 3, 2, 1, 14, 1, 10, 15, 1, 2, 1, 3, 5, 1, 4, 1, 2, 6, 1, 7, 1, 3, 2, 1, 9, 1, 12, 8, 1, 2, 1, 3, 4, 1, 5, 1, 2, 11, 1, 6, 1, 3, 2, 1, 10, 1, 4, 13, 1, 2, 1, 3, 5, 1, 7, 1, 2, 8, 1, 9, 1, 3, 2, 1, 4, 1, 6, 14, 1, 2, 1, 3, 12, 1, 5, 1, 2, 15, 1, 4, 1, 3, 2, 1, 7, 1, 10, 8, 1, 2, 1, 3, 5, 1, 4, 1, 2, 6, 1, 9, 1, 3, 2, 1, 11, 1, 13, 16, 1, 2, 1, 3, 4, 1, 5, 1, 2, 7, 1, 6, 1, 3, 2, 1, 8, 1, 4, 10, 1, 2, 1, 3, 5, 1, 9, 1, 2, 12, 1, 14, 1, 3, 2, 1, 4, 1, 6, 7, 1, 2, 1, 3, 11, 1, 5, 1, 2, 8, 1, 4, 1, 3, 2, 1, 13, 1, 10, 15, 1, 2, 1, 3, 5, 1, 4, 1, 2, 6, 1, 7, 1, 3, 2, 1, 9, 1, 12, 8, 1, 2, 1, 3, 4, 1, 5, 1, 2, 11, 1, 6, 1, 3, 2, 1, 10, 1, 4, 14, 1, 2, 1, 3, 5, 1, 7, 1, 2, 8, 1, 9, 1, 3, 2, 1, 4, 1, 6, 13, 1, 2, 1, 3, 12, 1, 5, 1, 2, 15, 1, 4, 1, 3, 2, 1, 7, 1, 10, 8, 1, 2, 1, 3, 5, 1, 4, 1, 2, 6, 1, 9, 1, 3, 2, 1, 11, 1, 14, 16

A periodic packing 13-coloring of $D(2,3)$ of period 240

1, 1, 2, 3, 4, 1, 1, 5, 6, 2, 1, 1, 8, 3, 13, 1, 1, 2, 4, 11, 1, 1, 7, 3, 2, 1, 1, 5, 6, 9, 1, 1, 2, 3, 4, 1, 1, 8, 10, 2, 1, 1, 12, 3, 5, 1, 1, 2, 4, 6, 1, 1, 7, 3, 2, 1, 1, 9, 11, 13, 1, 1, 2, 3, 4, 1, 1, 5, 6, 2, 1, 1, 8, 3, 7, 1, 1, 2, 4, 10, 1, 1, 12, 3, 2, 1, 1, 5, 6, 9, 1, 1, 2, 3, 4, 1, 1, 7, 8, 2, 1, 1, 11, 3, 5, 1, 1, 2, 4, 6, 1, 1, 10, 3, 2, 1, 1, 9, 13, 7, 1, 1, 2, 3, 4, 1, 1, 5, 6, 2, 1, 1, 8, 3, 12, 1, 1, 2, 4, 11, 1, 1, 7, 3, 2, 1, 1, 5, 6, 9, 1, 1, 2, 3, 4, 1, 1, 8, 10, 2, 1, 1, 13, 3, 5, 1, 1, 2, 4, 6, 1, 1, 7, 3, 2, 1, 1, 9, 11, 12, 1, 1, 2, 3, 4, 1, 1, 5, 6, 2, 1, 1, 8, 3, 7, 1, 1, 2, 4, 10, 1, 1, 13, 3, 2, 1, 1, 5, 6, 9, 1, 1, 2, 3, 4, 1, 1, 7, 8, 2, 1, 1, 11, 3, 5, 1, 1, 2, 4, 6, 1, 1, 10, 3, 2, 1, 1, 9, 12, 7

A periodic packing 23-coloring of $D(2,5)$ of period 336

1, 1, 2, 2, 1, 3, 4, 1, 1, 5, 6, 1, 7, 8, 1, 1, 2, 2, 1, 3, 10, 1, 1, 11, 4, 1, 15, 12, 1, 1, 2, 2, 1, 3, 16, 1, 1, 5, 6, 1, 4, 9, 1, 1, 2, 2, 1, 3, 7, 1, 1, 8, 14, 1, 17, 13, 1, 1, 2, 2, 1, 3, 4, 1, 1, 5, 6, 1, 10, 19, 1, 1, 2, 2, 1, 3, 11, 1, 1, 7, 4, 1, 9, 12, 1, 1, 2, 2, 1, 3, 8, 1, 1, 5, 6, 1, 4, 15, 1, 1, 2, 2, 1, 3, 18, 1, 1, 20, 21, 1, 7, 22, 1, 1, 2, 2, 1, 3, 4, 1, 1, 5, 6, 1, 10, 9, 1, 1, 2, 2, 1, 3, 8, 1, 1, 11, 4, 1, 13, 12, 1, 1, 2, 2, 1, 3, 7, 1, 1, 5, 6, 1, 4, 14, 1, 1, 2, 2, 1, 3, 16, 1, 1, 17, 23, 1, 9, 19, 1, 1, 2, 2, 1, 3, 4, 1, 1, 5, 6, 1, 7, 8, 1, 1, 2, 2, 1, 3, 10, 1 , 1, 11, 4, 1, 15, 12, 1, 1, 2, 2, 1, 3, 13, 1, 1, 5, 6, 1, 4, 9, 1, 1, 2, 2, 1, 3, 7, 1, 1, 8, 18, 1, 14, 20, 1, 1, 2, 2, 1, 3, 4, 1, 1, 5, 6, 1, 10, 21, 1, 1, 2, 2, 1, 3, 11, 1, 1, 7, 4, 1, 9, 12, 1, 1, 2, 2, 1, 3, 8, 1, 1, 5, 6, 1, 4, 13, 1, 1, 2, 2, 1, 3, 15, 1, 1, 16, 17, 1, 7, 19, 1, 1, 2, 2, 1, 3, 4, 1, 1, 5, 6, 1, 10, 9, 1, 1, 2, 2, 1, 3, 8, 1, 1, 11, 4, 1, 14, 12, 1, 1, 2, 2, 1, 3, 7, 1, 1, 5, 6, 1, 4, 18, 1, 1, 2, 2, 1, 3, 13, 1, 1, 20, 22, 1, 9, 23

B. An illustration of Subcase A.3 of the proof of Proposition 1

We illustrate the construction of a packing coloring of $D(1,t)$ defined in Subcase A.3 for $t = 75$ or $t = 77$, i.e. $s = 76 = 48 + 28$.

The color patterns R_j^i are defined as follows:

$R_1^i = (1,4,1,5,1,8,1,4,1,5,1,9)^6.(1,32+i,1,35+i)$, $0 \le i \le 2$;

$R_2^i = (1,4,1,5,1,10,1,4,1,5,1,11)^6.(1,38+i,1,41+i)$, $0 \le i \le 2$;

$R_3^i = (1,6,1,7,1,12,1,13,1,6,1,7,1,14,1,15)^4.(1,6,1,7,1,44+i,1,48+i,1,52+i,1,56+i)$, $0 \le i \le 3$;

$R_4^i = (1,6,1,7,1,16,1,17,1,6,1,7,1,18,1,19,1,6,1,7,1,20,1,21)^3.$
$(1,60+i,1,62+i), 0 \le i \le 1$;

$R_5^i = (1,6,1,7,1,22,1,23,1,6,1,7,1,24,1,25,1,6,1,7,1,26,1,27,1,6,1,7,1,28,1,29).$
$(1,6,1,7,1,64+i,1,67+i,1,70+i,1,73+i)$, $0 \le i \le 2$;

And a packing 75-coloring is obtained by assigning to subgraphs B_{2i+1} the color pattern $(1,2,1,3)^{19}$ and repeatedly to 48 consecutive subgraphs B_{2i} the sequence of color patterns

$$\mathcal{R} = (R_1^0, R_3^0, R_2^0, R_4^0, R_1^1, R_3^1, R_2^1, R_5^0, R_1^2, R_3^2, R_2^2, R_4^1, R_1^0, R_3^3, R_2^0, R_5^1, R_1^1, R_3^0, R_2^1, R_4^0, R_1^2, R_3^1, R_2^2, R_5^2,$$
$$R_1^0, R_3^2, R_2^0, R_4^1, R_1^1, R_3^3, R_2^1, R_5^0, R_1^2, R_3^0, R_2^2, R_4^0, R_1^0, R_3^1, R_2^0, R_5^1, R_1^1, R_3^2, R_2^1, R_4^1, R_1^2, R_3^3, R_2^2, R_5^2).$$