Radio Labelings of Distance Graphs

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Extended Abstract

A *radio k-labeling* of a connected graph G is an assignment f of non negative integers to the vertices of G such that

$$|f(x) - f(y)| \geq k + 1 - d(x, y),$$

for any two distinct vertices x and y, where $d(x, y)$ is the distance between x and y in G. The *radio k-labeling number* $\mathrm{rl}_k(G)$ of G is the minimum of $\max_{x,y \in V(G)} |f(x) - f(y)|$ over all radio k-labelings f of G.

The study of radio k-labelings was initiated by Chartrand et al. [1], motivated by radio channel assignment problems with interference constraints.

Except for paths [1, 3] and cycles [5], radio k-labelings have been investigated mainly for fixed values of k. This problem generalizes both the classical proper vertex-colouring problem (when $k = 1$) and the the well studied $L(2, 1)$-labeling problem (when $k = 2$). The other values of k considered were when k is close to the diameter of the graph. The interested reader is referred to surveys [2, 7] and recent papers [6, 8] for complementary results.

For a set of positive integers $\{d_1, d_2, \ldots, d_t\}$, the (infinite) distance graph $D(d_1, d_2, \ldots, d_t)$ has the set $\mathbb{Z}$ of integers as vertex set, with two distinct vertices $i, j \in \mathbb{Z}$ being adjacent if and only if $|i - j| = d_\ell$, for some ℓ.

Concerning radio k-labelings of distance graphs, the only known results are for $k = 2$ and mainly for 4-regular distance graphs [4, 9]. Moreover, for the path P_n of order n (a finite subgraph of $D(1) = P_\infty$), the following bounds were proved in [1, 3]: for any $n > 3$ and any $1 \leq k \leq n - 3$,

$$\frac{k^2+4}{2} \leq \mathrm{rl}_k(P_n) \leq \frac{k^2+2k}{2}, \text{ if } k \text{ is even,}$$

$$\frac{k^2+1}{2} \leq \mathrm{rl}_k(P_n) \leq \frac{k^2+2k-1}{2} \text{ if } k \text{ is odd;}$$

and it was conjectured in [3] that the upper bound is the exact value of the radio k-labeling number when the length of the path is large enough.

We prove the following results :

$$\tfrac{t}{2}k^2 + \tfrac{1}{2} \leq \mathrm{rl}_k(D(1, 2, \ldots, t)) \leq \begin{cases} \tfrac{t}{2}k^2 + \tfrac{t}{2}k, & \text{when } k \text{ is odd,} \\ \tfrac{t}{2}k^2 + k, & \text{when } k \text{ is even.} \end{cases}$$

$$\begin{array}{lclcll} \tfrac{t}{2}k^2 - P_2(t)k + P_3(t) & \leq & \mathrm{rl}_k(D(1,t)) & \leq & \tfrac{t}{2}k^2, & \text{for } t \geq 3 \text{ and odd } k, \\ \tfrac{t}{2}k^2 - Q_2(t)k + Q_3(t) & \leq & \mathrm{rl}_k(D(t-1,t)) & \leq & \tfrac{t}{2}k^2 + k - \tfrac{t+2}{2}, & \text{for } t \geq 3 \text{ and odd } k. \end{array}$$

where $P_i(t)$ and $Q_i(t)$ denote polynomials of variable t of degree i.

For each upper bound, we have found a corresponding coloring sequence with the desired number of labels while lower bounds were obtained by bounding the upper traceable number of the distance graphs by a function of the same parameter on the infinite path.

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