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# The Packing Coloring of Distance Graphs $D(k, t)$

Jan Ekstein\*    Přemysl Holub\*    Olivier Togni†

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## Abstract

The packing chromatic number  $\chi_\rho(G)$  of a graph  $G$  is the smallest integer  $p$  such that vertices of  $G$  can be partitioned into disjoint classes  $X_1, \dots, X_p$  where vertices in  $X_i$  have pairwise distance greater than  $i$ . For  $k < t$  we study the packing chromatic number of infinite distance graphs  $D(k, t)$ , i.e. graphs with the set  $\mathbb{Z}$  of integers as vertex set and in which two distinct vertices  $i, j \in \mathbb{Z}$  are adjacent if and only if  $|i - j| \in \{k, t\}$ .

We generalize results by Ekstein et al. for graphs  $D(1, t)$ . For sufficiently large  $t$  we prove that  $\chi_\rho(D(k, t)) \leq 30$  for both  $k, t$  odd, and that  $\chi_\rho(D(k, t)) \leq 56$  for exactly one of  $k, t$  odd. We also give some upper and lower bounds for  $\chi_\rho(D(k, t))$  with small  $k$  and  $t$ .

**Keywords:** distance graph; packing coloring; packing chromatic number

**AMS Subject Classification (2010):** 05C12, 05C15

## 1 Introduction

The concept of a packing coloring was introduced by Goddard et al. [9] under the name broadcast coloring where an application to frequency assignments

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was indicated. In a given network the signals of two stations that are using the same broadcast frequency will interfere unless they are located sufficiently far apart. The distance, in which the signals will propagate, is directly related to the power of those signals. Brešar et al. in [2] mentioned that this concept could have several additional applications, as, for instance, in resource placements and biological diversity (different species in a certain area require different amounts of territory). Moreover, the concept is both a packing and a coloring (i.e., a partitioning) concept. Therefore Brešar et al. in [2] proposed a notion *packing coloring* which we follow in this paper.

In this paper we consider simple undirected graphs only. For terminology and notations not defined here we refer to [1]. Let  $G$  be a connected graph and let  $\text{dist}_G(u, v)$  denote the distance between vertices  $u$  and  $v$  in  $G$ . We ask for a partition of the vertex set of  $G$  into disjoint classes  $X_1, \dots, X_p$  according to the following constraints. Each color class  $X_i$  should be an *i-packing*, a set of vertices with property that any distinct pair  $u, v \in X_i$  satisfies  $\text{dist}_G(u, v) > i$ . Such a partition is called a *packing p-coloring*, even though it is allowed that some sets  $X_i$  may be empty. The smallest integer  $p$  for which there exists a packing  $p$ -coloring of  $G$  is called the *packing chromatic number* of  $G$  and it is denoted  $\chi_\rho(G)$ . The determination of the packing chromatic number is computationally difficult. It was shown to be  $\mathcal{NP}$ -complete for general graphs in [9]. Fiala and Golovach [7] showed that the problem remains  $\mathcal{NP}$ -complete even for trees.

Let  $D = \{d_1, d_2, \dots, d_k\}$ , where  $d_i$  ( $i = 1, 2, \dots, k$ ) are positive integers such that  $d_1 < d_2 < \dots < d_k$ . The (infinite) *distance graph*  $D(d_1, d_2, \dots, d_k)$  has the set  $\mathbb{Z}$  of integers as a vertex set and in which two distinct vertices  $i, j \in \mathbb{Z}$  are adjacent if and only if  $|i - j| \in D$ . The study of a coloring of distance graphs was initiated by Eggleton et al. [6] and a lot of papers concerning this topic have been published (see [3], [11], [12], [13], [15] for a sample of results).

The study of a packing coloring of distance graphs was initiated by Togni. In [14] Togni showed that  $\chi_\rho(D(1, t)) \leq 40$  for odd  $t \geq 447$  and that  $\chi_\rho(D(1, t)) \leq 81$  for even  $t \geq 448$ . Ekstein et al. in [5] improved these upper bounds and proved that  $\chi_\rho(D(1, t)) \leq 35$  for odd  $t \geq 575$  and that  $\chi_\rho(D(1, t)) \leq 56$  for even  $t \geq 648$ . A lower bound 12 for the packing chromatic number of  $D(1, t)$ , for  $t \geq 9$ , is also given in [5].

In this paper we generalize mentioned results as follows.

**Theorem 1.** *Let  $k, t$  be odd positive integers such that  $t \geq 825$  and  $k, t$  coprime (i. e.  $D(k, t)$  is a connected distance graph). Then*

$$\chi_\rho(D(k, t)) \leq 30.$$

Note that, for  $k = 1$ , Theorem 1 also improves the upper bound given in [5].

**Corollary 2.** *For any odd positive integer  $t \geq 825$ ,  $\chi_\rho(D(1, t)) \leq 30$ .*

**Theorem 3.** *Let  $k, t$  be positive integers such that  $k$  is odd,  $t \geq 898$  is even and  $k, t$  coprime (i.e.  $D(k, t)$  is a connected distance graph). Then*

$$\chi_\rho(D(k, t)) \leq 56.$$

**Theorem 4.** *Let  $k, t$  be positive integers such that  $k$  is even,  $t \geq 923$  is odd and  $k, t$  coprime (i. e.  $D(k, t)$  is a connected distance graph). Then*

$$\chi_\rho(D(k, t)) \leq 56.$$

**Theorem 5.** *Let  $D(k, t)$  be a connected distance graph,  $t \geq 9$ . Then*

$$\chi_\rho(D(k, t)) \geq 12.$$

For  $k, t$  both even and also for  $k, t$  commensurable, the distance graph  $D(k, t)$  is disconnected and contains copies of a distance graph  $D(k', t')$  as its components with  $k' < k, t' < t$ , at least one of  $k', t'$  odd and  $k', t'$  coprime (as will be shown in a proof of Lemma 8). In the view of this fact, we can color each copy of  $D(k', t')$  in the same way, thus we obtain the following statement.

**Proposition 6.** *Let  $k, t$  be positive integers and  $g$  their greatest common divisor. Then  $\chi_\rho(D(k, t)) = \chi_\rho\left(D\left(\frac{k}{g}, \frac{t}{g}\right)\right)$ .*

For small values of  $k, t$  we give lower and upper bounds as it is shown in the following Table 1. Note that, for  $k = 1$ , an analogous table was published in [5].

Throughout the rest of the paper by a coloring we mean a packing coloring.

$k \setminus t$	3	4	5	6	7	8	9	10
2	<b>13</b>	$D(1, 2)$	14 – 22	$D(1, 3)$	15 – 27	$D(1, 4)$	12 – 31	$D(1, 5)$
3	—	14 – 19	<b>13</b>	$D(1, 2)$	13 – 17	14 – 28	$D(1, 3)$	13 – 29
4	—	—	13 – 22	$D(2, 3)$	16 – 32	$D(1, 2)$	15 – 32	$D(2, 5)$
5	—	—	—	15 – 29	13 – 20	14 – 32	13 – 23	$D(1, 2)$
6	—	—	—	—	15 – 29	$D(3, 4)$	$D(2, 3)$	$D(3, 5)$
7	—	—	—	—	—	14 – 34	12 – 23	12 – 40
8	—	—	—	—	—	—	12 – 37	$D(4, 5)$
9	—	—	—	—	—	—	—	12 – 42

Table 1: Values and bounds of  $\chi_\rho(D(k, t))$  for  $2 \leq k < t \leq 10$ . The emphasized numbers are exact values and all pairs of values are lower and upper bounds.

## 2 Bounds for $\chi_\rho(D(k, t))$ for small $k, t$

In this section we determine new lower and upper bounds for the packing chromatic number of  $D(k, t)$  which are mentioned in Table 1.

For the upper bounds, we found and verified (with a help of a computer) patterns, which can be periodically repeated for a whole distance graph  $D(k, t)$ . This means that we color vertices  $1, \dots, l$  of  $D(k, t)$  using a pattern of length  $l$  and copy this pattern on vertices  $1 + pl, \dots, l + pl, p \in \mathbb{Z}$ . As most of these patterns are of big lengths, they do not appear in this paper, but can be viewed at the web page <http://le2i.cnrs.fr/o.togni/packdist/> and tested using the java applet provider.

$k, t$	$c$	$p$	Configurations	Time
2, 3	12	213	$1.1 \cdot 10^{12}$	46 hours
2, 5	13	45	$5.9 \cdot 10^{12}$	327 hours
3, 4	13	43	$5.6 \cdot 10^{12}$	297 hours
3, 5	12	106	$5.7 \cdot 10^{11}$	35 hours
3, 7	12	54	$2 \cdot 10^{12}$	179 hours
4, 5	12	37	$4 \cdot 10^{11}$	23 hours

Table 2: Computations for finding lower bounds of  $\chi_\rho(D(k, t))$ . Time of the computation is measured on a one-core workstation from year 2012.

$k, t$	$q$	$b$	Configurations	Time
2, 7	5	37/45	$32.7 \cdot 10^9$	14 min
3, 8	7	38/42	$1.5 \cdot 10^{11}$	1 hours
3, 10	8	47/50	$10^{13}$	125 hours
4, 7	7	44/50	$7.6 \cdot 10^{12}$	58 hours
4, 9	7	47/52	$1.9 \cdot 10^{13}$	145 hours
5, 6	6	43/50	$1.1 \cdot 10^{12}$	7 hours
5, 7	8	47/50	$2.7 \cdot 10^{12}$	19 hours
5, 8	8	42/45	$7.8 \cdot 10^{12}$	58 hours
5, 9	7	48/52	$1.2 \cdot 10^{13}$	107 hours
6, 7	6	44/50	$3.5 \cdot 10^{12}$	27 hours
7, 8	6	50/55	$1.4 \cdot 10^{13}$	120 hours

Table 3: Computations for finding lower bounds of  $\chi_\rho(D(k, t))$ . Time of the computation is measured on a one-core workstation from year 2012.

For the lower bounds, we followed methods used in proofs of Lemmas 6 and 8 in [5]. Some of the lower bounds were obtained using brute force search programs (one in Pascal and one in C++). We showed that a subgraph  $D_p(k, t)$  of  $D(k, t)$  induced by vertices  $\{1, 2, \dots, p\}$  cannot be colored using colors from 1 to  $c$ , for some  $p$  and  $c$  (the results of computations can be seen in Table 2), which implies that  $\chi_\rho(D(k, t)) \geq c + 1$ . For a shortening of a computation time the programs precolored vertex 1 with color  $c$  and tried to extend the coloring for whole  $D_p(k, t)$ .

For the remaining lower bounds we used a density method. A density of a color class  $X_i$  in a packing coloring of  $G$  can be defined as a fraction of all vertices of  $X_i$  and all vertices of  $G$ . For the exact definition of densities  $d(X_i)$  and  $d(X_1 \cup \dots \cup X_i)$  we refer to [5].

The density method is based on the following proposition.

**Lemma 7.** [8] *For every finite packing coloring with  $k$  classes  $X_1, X_2, \dots, X_k$  of a graph  $G$  and any positive integer  $l$  satisfying  $1 \leq l \leq k$ , it holds that*

$$\sum_{i=1}^k d(X_i) \geq d(X_1 \cup \dots \cup X_l) + \sum_{i=l+1}^k d(X_i) \geq d(X_1 \cup \dots \cup X_k) = 1.$$

Again using brute force search programs (one in Pascal and one in C++) we showed that the density  $d(X_1 \cup X_2 \cup \dots \cup X_q)$  is bounded by  $b$ , for some  $q$  and  $b$  (the results are summarized in Table 3). For instance, for  $k, t = 2, 7$  we found that  $d(1, 2, 3, 4) \leq 32/41$ . Since there is no pair of vertices in  $D_{7i-13}(2, 7)$  with distance greater than  $i$  ( $i \geq 5$ ),  $d(i) = 1/(7i - 13)$  for  $i \geq 5$ . Then  $d(1, 2, 3, 4) + d(5) + \dots + d(13) \leq 0.9915326 < 1$  and hence, from Lemma 7,  $\chi_\rho(D(2, 7)) \geq 14$ . The bounds for the other values of  $k, t$  in Table 3 are proved similarly, observing that for  $D(k, t)$ ,  $d(i) = 1/(ti - \alpha)$ , with  $\alpha = 13$  for  $k, t = 2, 7$  and  $i \geq 5$ ;  $\alpha = 17$  for  $k, t = 3, 8$  and  $i \geq 8$ ;  $\alpha = 22$  for  $k, t = 3, 10$  and  $i \geq 9$ ;  $\alpha = 8$  for  $k, t = 4, 7$  and  $i \geq 8$ ;  $\alpha = 21$  for  $k, t = 4, 9$  and  $i \geq 8$ ;  $\alpha = 5$  for  $k, t = 5, 7$  and  $i \geq 9$ ;  $\alpha = 10$  for  $k, t = 5, 8$  and  $i \geq 9$ ;  $\alpha = 19$  for  $k, t = 5, 9$  and  $i \geq 8$ ; and  $\alpha = -1$  for  $k = t - 1$  and  $i \geq t - 1$ .

The lower bounds for  $D(2, 9)$ ,  $D(7, 9)$ ,  $D(7, 10)$ ,  $D(8, 9)$  and  $D(9, 10)$  are obtained from Theorem 5.

### 3 Proofs

First of all we prove for which  $k, t$  a distance graph  $D(k, t)$  is connected.

**Lemma 8.** *A distance graph  $D(k, t)$  is connected if and only if the greatest common divisor of  $k, t$  is 1.*

*Proof.* If the greatest common divisor of  $k, t$  is 1, then from linear algebra  $1 = mk + nt$ , where  $m, n \in \mathbb{Z}$ . For every vertex corresponding to a number  $p \in \mathbb{Z}$  it holds that  $p = pmk + pnt$  and therefore there exists a path between this vertex and vertex corresponding to 0. Hence it follows that there exists a path between any pair of vertices and that  $D(k, t)$  is connected.

If the greatest common divisor of  $k, t$  is  $p > 1$ , then we divide a set of vertices of  $D(k, t)$  into subsets in terms of equivalence classes modulo  $p$  and clearly there is no edge between any pair of vertices from distinct subsets of vertices of  $D(k, t)$ . Hence  $D(k, t)$  is not connected which completes the proof. Moreover, vertices corresponding to the equivalence classes modulo  $p$  induce  $p$  isomorphic copies of a graph  $D(\frac{k}{p}, \frac{t}{p})$  where  $\frac{k}{p}, \frac{t}{p}$  are coprime positive integers.  $\square$

A key observation of this section is that a connected distance graph  $D(k, t)$  can be drawn as  $k$  vertex disjoint infinite spirals with  $t$  lines orthogonal to the spirals (e.g.  $D(3, 8)$  on Fig. 1).

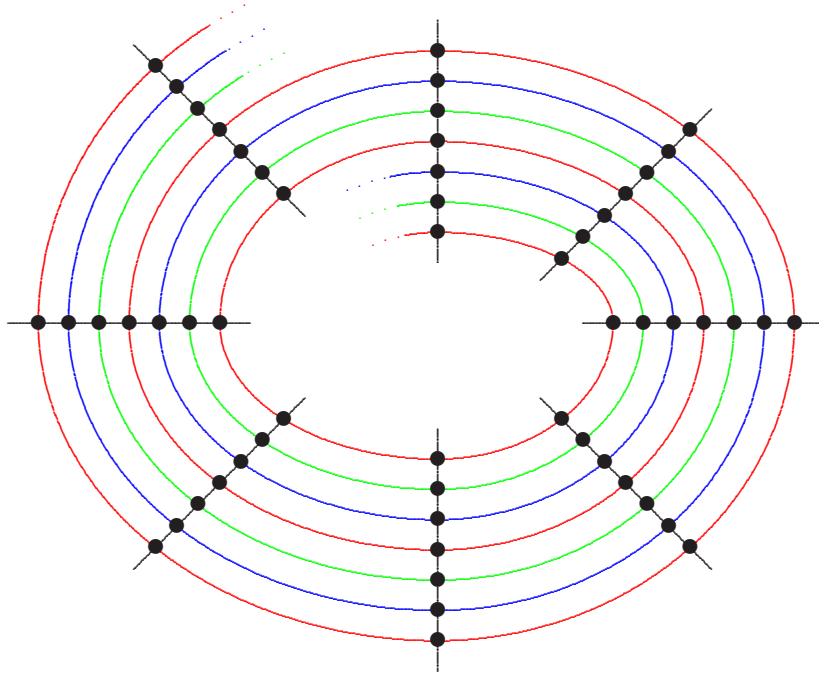


Figure 1: Distance graph  $D(3, 8)$ .

For  $i \in \{0, 1, \dots, t-1\}$ , the  $i$ -band in a connected distance graph  $D(k, t)$ , denoted by  $B_i$ , is an infinite path in  $D(k, t)$  on the vertices  $V(B_i) = \{ik + jt, j \in \mathbb{Z}\}$ . For  $i \in \{0, 1, \dots, t-24\}$ , the  $i$ -strip in a connected distance graph  $D(k, t)$ ,  $t > 24$ , denoted by  $S_i$ , is a subgraph of  $D(k, t)$  induced by the union of vertices of  $B_i, B_{i+1}, \dots, B_{i+23}$ .

For a connected graph  $D(k, t)$  we use a notation  $D(k, t) = S_0 B_{24} S_{25} B_{49} \dots$  to express that we view  $D(k, t)$  as a union of strips  $S_0, S_{25}, \dots$  and bands  $B_{24}, B_{49}, \dots$  (including edges between strips and bands).

It is obvious that an  $i$ -strip in  $D(k, t)$  is isomorphic to an  $i$ -strip in  $D(1, t)$ . Hence we apply results in [5] for a coloring of strips in  $D(k, t)$ . Moreover we color vertices of all strips using the pattern on  $24 \times 24$  vertices made by Holub and Soukal in [10], in which it is possible to replace color 16 (17) by 22 (23), because two vertices colored with 16 (17) are in a whole strip at distance at least 24, respectively (see Fig. 2).

**Lemma 9.** *Let  $D(k, t)$  be a connected distance graph,  $t > 24$ , and  $S_i$  its  $i$ -strip. Then it is possible to color  $S_i$  using colors  $C = \{1, 2, \dots, 14, 15, 22, 23\}$ .*

Then we can use colors 16 and 17 for coloring of bands as it is explained in the following statement.



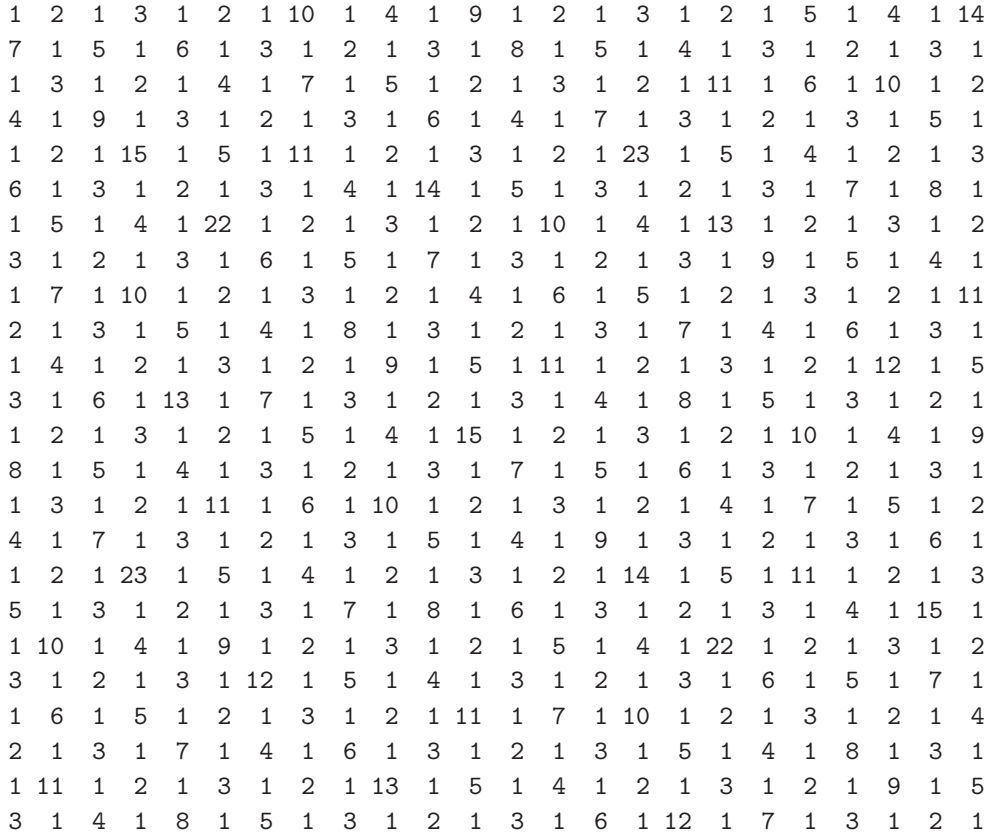


Figure 2: A modified pattern on  $24 \times 24$  vertices.

**Lemma 10.** *Let  $D(k, t)$  be a connected distance graph,  $t \geq 56$ , and  $B_i, B_{i+25}$  its bands. Then it is possible to color  $B_i$  and  $B_{i+25}$  using colors  $C = \{1, 16, 17, \dots, 21, 24, 25, \dots, 30\}$ .*

*Proof.* We prove this lemma by exhibiting a repeating pattern using colors 1, 16, 17, ..., 21, 24, 25, ..., 30. The pattern was found with help of a computer, it has period 144 and is given here:

1, 16, 1, 19, 1, 24, 1, 17, 1, 26, 1, 25, 1, 18, 1, 20, 1, 21, 1, 16, 1, 27, 1, 19,  
 1, 17, 1, 28, 1, 29, 1, 24, 1, 18, 1, 30, 1, 16, 1, 20, 1, 21, 1, 17, 1, 19, 1, 25,  
 1, 26, 1, 27, 1, 18, 1, 16, 1, 24, 1, 28, 1, 17, 1, 20, 1, 19, 1, 21, 1, 29, 1, 30,  
 1, 16, 1, 18, 1, 25, 1, 17, 1, 26, 1, 24, 1, 19, 1, 20, 1, 21, 1, 16, 1, 27, 1, 18, (\*)  
 1, 17, 1, 28, 1, 29, 1, 25, 1, 19, 1, 30, 1, 16, 1, 20, 1, 21, 1, 17, 1, 18, 1, 24,  
 1, 26, 1, 27, 1, 19, 1, 16, 1, 25, 1, 28, 1, 17, 1, 20, 1, 18, 1, 21, 1, 29, 1, 30.

We color  $B_i$  cyclically with pattern  $(*)$  starting at the vertex  $i$  and  $B_{i+25}$  cyclically with pattern  $(*)$  starting at the vertex  $(i + 25)k + jt$  for any  $j \in \{-25, -24, \dots, -6, 6, 7, \dots, 25\}$ . Let  $D_c$  be a minimal distance between two vertices colored with the same color  $c$  in the same band. Then from the coloring of a whole band with pattern  $(*)$  we have  $D_{25} = 26$ ,  $D_{26} = 32$ ,  $D_{27} = 30$ ,  $D_{28} = 32$ ,  $D_{29} = 32$  and  $D_{30} = 36$ . Let  $u \in V(B_i)$  and  $v \in V(B_{i+25})$  be colored with the same color  $c$ . For  $c \leq 24$ , the distance between  $u$  and  $v$  is greater than 24. For  $c \geq 25$ , the distance between  $u$  and  $v$  is at least  $\min\{|j|, D_c - |j|\} + 25$  which is greater than  $c$ . Hence we have a packing coloring of  $B_i$  and  $B_{i+25}$ .  $\square$

In proofs of Theorems 3 and 4 we use the following statement proved by Goddard et al. in [9].

**Proposition 11.** [9] *For every  $l \in \mathbb{N}$ , the infinite path can be colored with colors  $l, l + 1, \dots, 3l + 2$ .*

Now we are ready to prove Theorems 1, 3 and 4.

### 3.1 Proof of Theorem 1

*Proof.* Let  $D_p = \{-25, -24, \dots, -6, 6, 7, \dots, 25\}$ . Let  $k_1 = \min\{k \pmod{24}, 24 - k \pmod{24}\}$ .

Let  $r, s$  be positive integers such that  $t = 24s + r$ , where  $r$  is odd (since  $t$  is odd) and minimal such that  $k_1 \leq r \leq 33$ . We prove Theorem 1 even for  $t \geq 24r + r$ , which is in the worst case (for  $r = 33$ ) the general bound  $t \geq 825$ . Hence  $s \geq r$  and we have  $s$  disjoint strips and  $r$  disjoint bands such that  $D(k, t) = S_0 B_{24} S_{25} B_{49} \dots S_{24(r-1)+r-1} B_{24r+r-1} S_{24r+r} \dots S_{24(s-1)+r}$ .

For odd  $i = 1, 3, \dots, r - k_1 + 1$ , we color the strips  $S_{24(i-1)+i-1}$  cyclically with the pattern from Fig. 2 starting at the vertex  $24(i-1)k + (i-1)k$ . If  $r > k_1$ , then, for even  $i = 2, 4, \dots, r - k_1$ , we color  $S_{24(i-1)+i-1}$  cyclically with the pattern from Fig. 2 starting at the vertex  $24(i-1)k + (i-1)k - t$ .

Let  $k_1 \equiv k \pmod{24}$ . Then, for  $i = r + 1, r + 2, \dots, s$ , we color  $S_{24(i-1)+r}$  cyclically with the pattern from Fig. 2 starting at the vertex  $24(i-1)k + rk - k_1 t$ . If  $k_1 \neq 1$ , then, for  $i = r - k_1 + 2, r - k_1 + 3, \dots, r$ , we color the strips

$S_{24(i-1)+i-1}$  cyclically with the pattern from Fig. 2 starting at the vertex  $24(i-1)k + (i-1)k - (i-r+k_1-1)t$ .

Let  $k_1 \equiv 24-k \pmod{24}$ . Then, for  $i = r+1, r+2, \dots, s$ , we color  $S_{24(i-1)+r}$  cyclically with the pattern from Fig. 2 starting at the vertex  $24(i-1)k + rk + k_1t$ . If  $k_1 \neq 1$ , then, for  $i = r-k_1+2, r-k_1+3, \dots, r$ , we color the strips  $S_{24(i-1)+i-1}$  cyclically with the pattern from Fig. 2 starting at the vertex  $24(i-1)k + (i-1)k + (i-r+k_1-1)t$ . Hence we have a packing coloring of all  $s$  disjoint strips of  $D(k, t)$  using Lemma 9.

For  $i = 1, 2, \dots, r$ , we color the bands  $B_{24i+i-1}$  cyclically with pattern (\*) starting at the vertex  $24ik + (i-1)k + j_it$  such that  $j_i$  is even (odd) for odd (even)  $i$ , respectively, for  $i > 1$ ,  $j_i - j_{i-1} \in D_p$ , and for  $s = r$ ,  $|j_r - j_1 + k| \pmod{144} \in D_p$ .

If  $s > r$ , then  $j_i$  exist. Now assume that  $r = s$ .

If  $r = s = 1$ , then we set  $j_1 = 0$ . Note that from  $k_1 \leq r$  we have only distance graphs  $D(1, 25)$  and  $D(23, 25)$ .

If  $r = s = 5$ , then we have distance graphs  $D(k, 125)$  with  $k \leq 123$  such that  $k_1 \leq r = 3$ . For  $k = 1, 3, 5, 19, 43, 45, 47, 49, 51$ , we set  $j_1 = 0$ ,  $j_2 = -13$ ,  $j_3 = -26$  and  $j_4, j_5$  such that  $j_4 - j_3 \in D_p$  and  $j_5 = j_3$ . For  $k = 21, 23, 25, 27, 29, 53, 67, 69, 71$ , we set  $j_1 = 0$ ,  $j_2 = -23$ ,  $j_3 = -46$  and  $j_4, j_5$  such that  $j_4 - j_3 \in D_p$  and  $j_5 = j_3$ . For  $k = 73, 75, 77, 91, 115, 117, 119, 121, 123$  we set  $j_1 = 0$ ,  $j_2 = -25$ ,  $j_3 = -50$ ,  $j_4 = -75$  and  $j_5 = -98$ . For  $k = 93, 95, 97, 99, 101$  we set  $j_1 = 0$ ,  $j_2 = -21$ ,  $j_3 = -42$ ,  $j_4 = -63$  and  $j_5 = -84$ .

If  $r = s = 3$ , then we have distance graphs  $D(k, 75)$  with  $k \leq 73$  such that  $k_1 \leq r = 3$  and we proceed for feasible  $k$  in a similar way as in the case  $r = s = 5$ .

If  $r = s \geq 7$ , then we proceed analogously to previous cases (a combination of  $r$  numbers from  $D_p$  could be from 0 to 144, which is the length of the pattern (\*)).

Hence we have a packing coloring of all  $r$  disjoint bands of  $D(k, t)$  using the same principle as in the proof of Lemma 10.

Note that the bands are colored with colors  $1, 16, 17, \dots, 21, 24, 25, \dots, 30$  and the strips are colored with colors  $1, 2, \dots, 15, 22, 23$  in such a way that no pair of adjacent vertices is colored with color 1. Then we conclude that we have a packing coloring of  $D(k, t)$ , hence  $\chi_\rho(D(k, t)) \leq 30$ .  $\square$

Some cases, in which we can decrease  $t$  for which Theorem 1 is true, are given in Table 4.

$r$	1	3	5	7	9	11	13	15	17
$k_1$	1	1, 3	1, 3, 5	1, 3, 5, 7	1, 3, 5, 7, 9	∇	∇	∇	∇
$t \geq$	25	75	125	175	225	275	325	375	425

$r$	19	21	23	25	27	29	31	33
$k_1$	∇	∇	∇	3, 5, 7, 9, 11	5, 7, 9, 11	7, 9, 11	9, 11	11
$t \geq$	475	525	575	625	675	725	775	825

Table 4: Table for  $t$  depending on odd  $k_1, r$  with  $r \geq k_1$ .

### 3.2 Proof of Theorem 3

*Proof.* Let  $D_p = \{-25, -24, \dots, -6, 6, 7, \dots, 25\}$ . Let  $k_1 = \min\{k \pmod{24}, 24 - k \pmod{24}\}$ .

Let  $r, s$  be positive integers such that  $t = 24(s + 2) + r$ , where  $r$  is even (since  $t$  is even) and minimal such that  $k_1 < r \leq 34$ . We prove Theorem 3 even for  $t \geq 24(r + 2) + r$ , which is in the worst case (for  $r = 34$ ) the general bound  $t \geq 898$ . Hence  $s \geq r$  and we have  $s + 2$  disjoint strips and  $r$  disjoint bands such that  $D(k, t) = S_0 S_{24} B_{48} S_{49} \dots S_{24(r-1)+r-2} B_{24r+r-2} S_{24r+r-1} S_{24(r+1)+r-1} \dots S_{24(s+1)+r-1} B_{24(s+2)+r-1}$ .

We color the strip  $S_0$  cyclically with the pattern from Fig. 2 starting at the vertex 0. For odd  $i = 1, 3, \dots, r - k_1$ , we color the strips  $S_{24i+i-1}$  cyclically with the pattern from Fig. 2 starting at the vertex  $24ik + (i - 1)k$ . If  $r > k_1 + 1$ , then, for even  $i = 2, 4, \dots, r - k_1 - 1$ , we color  $S_{24i+i-1}$  cyclically with the pattern from Fig. 2 starting at the vertex  $24ik + (i - 1)k - t$ .

Let  $k_1 \equiv k \pmod{24}$ . Then, for  $i = r + 1, \dots, s + 1$ , we color  $S_{24i+r-1}$  cyclically with the pattern from Fig. 2 starting at the vertex  $24ik + (r - 1)k - k_1 t$ . If  $k_1 \neq 1$ , then, for  $i = r - k_1 + 1, r - k_1 + 2, \dots, r$ , we color the strips  $S_{24i+i-1}$  cyclically with the pattern from Fig. 2 starting at the vertex  $24ik + (i - 1)k - (i - r + k_1)t$ .

Let  $k_1 \equiv 24 - k \pmod{24}$ . Then, for  $i = r + 1, r + 2, \dots, s + 1$ , we color  $S_{24i+r-1}$  cyclically with the pattern from Fig. 2 starting at the vertex

$24ik + (r - 1)k + k_1t$ . If  $k_1 \neq 1$ , then, for  $i = r - k_1 + 1, r - k_1 + 2, \dots, r$ , we color the strips  $S_{24i+i-1}$  cyclically with the pattern from Fig. 2 starting at the vertex  $24ik + (i - 1)k + (i - r + k_1)t$ . Hence we have a packing coloring of all  $s$  disjoint strips of  $D(k, t)$  using Lemma 9.

For  $i = 1, 2, \dots, r - 1$ , we color the bands  $B_{24(i+1)+i-1}$  cyclically with pattern (\*) starting at the vertex  $24(i + 1)k + (i - 1)k + j_it$  such that  $j_i$  is even (odd) for odd (even)  $i$ , respectively, and for  $i > 1$ ,  $j_i - j_{i-1} \in D_p$ .

We color  $B_{24(s+2)+r-1}$  with a sequence of colors 18, 19, ..., 21, 16, 17, 24, 25, ..., 56 starting at any vertex of  $B_{24(s+2)+r-1}$ . By Proposition 11 for  $l = 18$ , we can color  $B_{24(s+2)+r-1}$  with colors 18, 19, ..., 56. Since we used colors 22 and 23 for a coloring of strips, we replace color 22 (23) by 16 (17), respectively, in the coloring of this band. Note the band  $B_{24(s+2)+r-1}$  is the only one with colors greater than 48. By Lemma 10 and the fact that a distance between any vertex of  $B_{24(s+2)+r-1}$  and any vertex of any other band of  $D(k, t)$  is at least 49 the defined coloring is a packing coloring of all  $r$  disjoint bands of  $D(k, t)$ .

Note that the bands are colored with colors 1, 16, 17, ..., 21, 24, 25, ..., 56 and the strips are colored with colors 1, 2, ..., 15, 22, 23 in such a way that no pair of adjacent vertices is colored with color 1. Then we conclude that we have a packing coloring of  $D(k, t)$ , hence  $\chi_\rho(D(k, t)) \leq 56$ .  $\square$

Some cases, in which we can decrease  $t$  for which Theorem 3 is true, are summarized in Table 5.

$r$	2	4	6	8	10	12	14	16	18
$k_1$	1	1, 3	1, 3, 5	1, 3, 5, 7	1, 3, 5, 7, 9	$\forall$	$\forall$	$\forall$	$\forall$
$t \geq$	98	148	198	248	298	348	398	448	498

$r$	20	22	24	26	28	30	32	34
$k_1$	$\forall$	$\forall$	$\forall$	3, 5, 7, 9, 11	5, 7, 9, 11	7, 9, 11	9, 11	11
$t \geq$	548	598	648	698	748	798	848	898

Table 5: Table for  $t$  depending on odd  $k_1$  and even  $r$  with  $r > k_1$ .

### 3.3 Proof of Theorem 4

*Proof.* Let  $D_p = \{-25, -24, \dots, -6, 6, 7, \dots, 25\}$ . Let  $k_1 = \min\{k \pmod{24}, 24 - k \pmod{24}\}$ .

Let  $s$  be a positive integer such that  $t = 24s + 1$  and  $k_1 = 0$ . Hence we have  $s$  disjoint strips and 1 band such that  $D(k, t) = S_0 S_{24} \dots S_{24(s-1)} B_{24s}$ . For  $i = 0, \dots, s-1$  we color the strips  $S_{24i}$  cyclically with the pattern from Fig. 2 starting at the vertex  $24ik$ . We color  $B_{24s}$  with a sequence of colors 18, 19, ..., 21, 16, 17, 24, 25, ..., 56 starting at any vertex of  $B_{24s}$ . Hence the band  $B_{24s}$  is colored with colors 1, 16, 17, ..., 21, 24, 25, ..., 56 and the strips are colored with colors 1, 2, ..., 15, 22, 23 in such a way that no pair of adjacent vertices is colored with color 1. Then we conclude that we have a packing coloring of  $D(k, t)$ , hence  $\chi_\rho(D(k, t)) \leq 56$ .

Let  $r, s$  be positive integers such that  $t = 24(s+2) + r$ , where  $r$  is odd (since  $t$  is odd) and minimal such that  $k_1 < r \leq 35$ . We exclude the previous case  $k_1 = 0, r = 1$  and we prove Theorem 4 even for  $t \geq 24(r+2) + r$ , which is in the worst case (for  $r = 35$ ) the general bound  $t \geq 923$ . Hence  $s \geq r$  and we have  $s+2$  disjoint strips and  $r$  disjoint bands such that  $D(k, t) = S_0 S_{24} B_{48} S_{48} \dots S_{24(r-1)+r-2} B_{24r+r-2} S_{24r+r-1} S_{24(r+1)+r-1} \dots S_{24(s+1)+r-1} B_{24(s+2)+r-1}$ .

The rest of the proof of Theorem 4 is exactly same as of the proof of Theorem 3.  $\square$

Some cases, in which we can decrease  $t$  for which Theorem 4 is true, are given in Table 6.

$r$	1	3	5	7	9	11	13	15	17	19
$k_1$	0	0, 2	0, 2, 4	0, 2, 4, 6	0, 2, 4, 6, 8	0, 2, 4, 6, 8, 10	$\forall$	$\forall$	$\forall$	$\forall$
$t \geq$	25	123	173	223	273	323	373	423	473	523

$r$	21	23	25	27	29	31	33	35
$k_1$	$\forall$	$\forall$	2, 4, 6, 8, 10, 12	4, 6, 8, 10, 12	6, 8, 10, 12	8, 10, 12	10, 12	12
$t \geq$	573	623	673	723	773	823	873	923

Table 6: Table for  $t$  depending on even  $k_1$  and odd  $r$  with  $r > k_1$ .

### 3.4 Proof of Theorem 5

*Proof.* It is shown in [4] that a finite square lattice  $15 \times 9$  cannot be colored using 11 colors. Clearly  $D(k, t)$  contains a finite square grid as a subgraph and  $t \geq 9$  assures existence of the square lattice  $15 \times 9$  in a connected  $D(k, t)$ . Therefore,  $\chi_\rho(D(k, t)) \geq 12$  for every  $t \geq 9$ .  $\square$

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