On the Complexity of the S-coloring problem

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December 18, 2013

Abstract

This work establishes the complexity class of several instances of the S-coloring problem: For a graph G, a positive integer k and a non decreasing list of integers $S = (s_1, \ldots, s_k)$, G admits a *S-coloring*, if its vertices can be partitioned into sets X_{s_i}, $i = 1, \ldots, k$, where each X_{s_i} being an s_i-packing (a set of vertices at pairwise distance greater than s_i). For a unfixed size of list, the complexity of the S-coloring problem is determined for several instances of the problem.

Keywords: NP-hard problem, distance, Packing chromatic number, d-distance coloring.

1 Introduction

We consider only undirected connected graphs in this paper. Given a graph $G = (V, E)$, an *i-packing* is a set $X_i \subseteq V(G)$ such that for any distinct pair u, $v \in X_i$, $d_G(u, v) > i$, where $d_G(u, v)$ denotes the usual shortest path distance between u and v. We will use X_i to refer to an i-packing in a graph G. For a non decreasing sequence of positive integers $S = (s_i, i \in \mathbb{N}^*)$, an *$S$-$k$-coloring* of G is a partition of $V(G)$ into sets $X_{s_1}, \ldots, X_{s_k}$, where each X_{s_i} being an s_i-packing. For a non decreasing list $S' = (s_1, \ldots, s_k)$ of k integers, an *S'-(packing)-coloring* of G is a S-k-coloring of G for a sequence S which begins with a list S'. A graph G is S-colorable if there exists a S-coloring of G. By $|S|$, we denote the size of the list S, and by s_i we denote the ith element of the list. Then we define the following decision problem:

S-COL
Instance : A graph G
Question: Does G have an S-coloring ?

*Author partially supported by the Burgundy Council

The problem $(1,\ldots,1)$-COL corresponds to the k-coloring problem (where k is the number of 1) which is known to be NP-complete for $k \geq 3$. The S-coloring problem generalizes coloring problems with distance constraints like the *packing chromatic number* or the *distance chromatic number* of a graph. The packing chromatic number [2] of G is the least integer k such that the vertex set of G can be partitioned into sets X_i, $i = 1,\ldots,k$, where each X_i being an i-packing. Hence, the least integer k such that G has a $(1,2,\ldots,k)$-coloring. The d-distance chromatic number [4] of G is the least integer k such that the vertex set of G can be partitioned into sets X_d^i, $i = 1,\ldots,k$, where each X_d^i being a d-packing. Initially, the S-coloring has been introduced by Goddard et al. [2] and Fiala et al. [1]. The S-coloring problem, with $|S| = 3$ has been studied by Goddard et al. [2] in order to determine the complexity of the packing chromatic number when $k = 4$. Moreover, Goddard and Xu [3] have proven that for $|S| = 3$, S-COL is NP-hard, if $s_1 = s_2 = 1$ or if $s_1 = 1$ and $s_2 = s_3 = 2$ and polynomial otherwise.
In order to represent an S-coloring, we will denote colors of vertices in different i-packings by i followed by a letter of the alphabet ($2a$ and $2b$ for example).

Remark 1.1. *Notice that the $(1,1)$-colorable graphs correspond to bipartite graphs. If $s_2 > 1$, only trees of diameter 2 are (s_1,s_2)-colorable. Hence, (s_1,s_2)-COL is polynomial time solvable.*

In the second section, properties on the S-coloring of paths are used to determine polynomial instances of S-COL. In the third section, various NP-hard instances of S-COL are determined.

2 S-coloring of path and polynomial instances of S-COL

2.1 S-coloring of infinite graphs

Proposition 2.1. *Let S be a non decreasing list of integers, with $s_1 > 1$ and G be a graph. If G contains a vertex of degree $|S|$, then G is not S-colorable.*

Proof. Let v be a vertex of degree $|S|$. As v and the neighbors of v are at pairwise distance at most 2, every vertex must have a different color and we only have $|S|$ colors. □

Definition 2.1. *Let k and d be two positive integers. By $n(k,d) = 1 + k\sum_{i=0}^{d-1}(k-1)^i$, we denote the Moore bound. The Moore bound corresponds to the maximum number of vertices for a graph of diameter d and maximal degree k.*

Proposition 2.2. *Let S be a non decreasing list of integers, with $s_1 > 1$. If an infinite graph is S-colorable, then P_∞ is S-colorable.*

Proof. Notice that a graph of diameter d has the graph P_d as subgraph. Hence, if an infinite graph with a unbounded diameter is S-colorable, then P_∞ is S-colorable. By contradiction, we suppose there exists an infinite S-colorable graph G with a fixed diameter d, $d > 0$. By Proposition 2.1, the maximum degree of G is $|S| - 1$. Therefore, using the Moore bound, G has order at most $n(|S| - 1, d)$. □

Definition 2.2. *Let S be a non decreasing list of integers. Let $t_{max}(S)$ be the order of the largest graph which admits an S-coloring. If $t_{max}(S)$ is not bounded, then we define $t_{max}(S)$ as ∞.*

Claim 2.3. *Let S be a non decreasing list of integers. If $t_{max}(S)$ is bounded, then there exists an S-colorable tree of order $t_{max}(S)$.*

Proposition 2.4. *Let S be a non decreasing list of integers. If $s_1 \geq |S|$ then $t_{max}(S) = |S|$.*

Proof. A graph of order $|S| + 1$ has diameter at most $|S|$. Hence, every vertex should be colored differently. □

2.2 S-coloring of the path

For a vertex $x \in V(G)$, the ball of radius n centered at x is the set $B_n(x) = \{v \in V(G) | d_G(x, v) \leq n\}$. The density of a set of vertices $X \subseteq V(G)$ is $d_G(X) = \limsup_{l \to \infty} \max_{x \in V} \{\frac{|X \cap B_l(x)|}{|B_l(x)|}\}$. Suppose P_∞ admits an S-coloring and let v be a vertex of P_∞ colored c. Let $N_c(v)$ be the set of vertices (no more than two vertices) which have the color c such that in the smallest path between each of these vertices and v, no vertex is colored c.

Proposition 2.5. *Let S be a non decreasing list of integers. For every S-coloring of P_∞ there exists another S-coloring such that for every vertex v, if v is colored s_i, for some i, then every vertex in $N_{s_i}(v)$ is at distance at most $2s_i - 1$ from v.*

Proof. Suppose that two vertices u and v colored s_i are such that $d(u, v) \geq 2s_i$ with no vertex colored s_i in the path between u and v. Let $l = \lfloor d(u, v)/s_i \rfloor$ and let $u_1, \dots u_{d(u,v)-1}$ be the vertices in the path between u and v. By adding a vertex between u and v, we mean remove the edge uv, add a vertex w and connect it to u and v. We add one vertex colored s_i between the vertices $u_{l's_i}$ and $u_{l's_i+1}$, for each l' such that $0 < l' < l$. We obtain an S-coloring such that for every vertex v colored s_i, every vertex in $N_{s_i}(v)$ is at distance at most $2s_i - 1$ from v. □

Corollary 2.6. *Let S be a non decreasing list of integers with $s_1 = 1$. For every S-coloring of P_∞ there exists another S-coloring such that for every vertex v colored 1, the vertices in $N_1(v)$ are at distance 2.*

Definition 2.3. *Let G be a graph. The maximum density of an i-packing in G is denoted $d_G(X_i)$.*

Claim 2.7. *Let S be a non decreasing list of integers. If $\sum_{i=1}^{|S|}(d_G(X_{s_i}) < 1\}$, then G is not S-colorable.*

Claim 2.8. *If $X_i \subseteq V(P_\infty)$ is an i-packing of P_∞, then $d(X_i) \leq \frac{1}{i+1}$.*

Definition 2.4. *Let S be a non decreasing list of integers. Let $p_{max}(S)$ be the order of the largest S-colorable path. If P_∞ is S-colorable, then we define $p_{max}(S)$ as ∞.*

Remark 2.9. *Let S be a non decreasing list of integers. First, remark that $p_{max}(S) \leq t_{max}(S)$. Second, if $p_{max}(S)$ is bounded and $s_1 > 2$, then using the Moore bound, we have $t_{max}(S) \leq n(|S|-1, p_{max}(S))$.*

Proposition 2.10. *Let S be a non decreasing list of integers. If $\sum_{i=1}^{|S|} \frac{1}{s_i+1} < 1$, then $p_{max}(S) < \infty$.*

Proof. Claim 2.8 allows to conclude that $\sum_{i=1}^{|S|} d(X_{s_i}) \leq \sum_{i=1}^{|S|} \frac{1}{s_i+1} < 1$. Thus, using claim 2.7 P_∞ does not admit an S-coloring. □

Proposition 2.11. *Let S be a non decreasing list of integers, with $s_1 = 1$. If $\sum_{i=1}^{|S|} \frac{1}{2\lceil (s_i-1)/2 \rceil + 2} < 1$, then $p_{max}(S) < \infty$.*

Proof. Suppose that $s_1 = 1$, Corollary 2.6 gives us that there exists an S-coloring where X_1, the set of vertices colored by 1, is one of the two independent sets that form a bipartition of P_∞. A consequence is that any i-packing, with i even, is an $(i+1)$-packing too. We obtain that if $\sum_{i=1}^{|S|} \frac{1}{2\lceil (s_i-1)/2 \rceil + 2} < 1$, then $p_{max}(S) < \infty$ (the formula corresponds to add one to even numbers). □

2.3 Polynomial instances of S-COL

Definition 2.5. *A Tree decomposition of a graph $G = (V, E)$ is a triple (T, F, X) consisting of an undirected tree (T, F) and a map $X : T \to 2^{|V|}$ associating to each vertex $i \in T$, a subset $X_i \subseteq V$ such that:*

1. *$V = \bigcup_{i \in T} X_i$;*
2. *For all edges $uv \in E$, there exists $i \in T$ such that u and v are in X_i;*
3. *If j lies on the path between i and k in (T, F) then $X_i \cap X_k \subseteq X_j$.*

The width of (T, F, X) is $max_{i \in T}(|X_i| - 1)$. The treewidth of G is the smallest width of a tree decomposition of G.

For a given list S such that $p_{max}(S) < \infty$, we aim to bound the treewidth of the S-colorable graphs, isolating vertices of degree at least k.

Proposition 2.12. *For a given S-colorable graph G with $p_{max}(S) < \infty$, the treewidth of G is bounded.*

Proof. If $s_1 > 1$ and $p_{max}(S) < \infty$, then by Proposition 2.2 the size of G is bounded. Thus, the treewidth of G is bounded. Suppose that $s_1 = 1$ and let X_1 be the set of vertices colored by 1 for a fixed S-coloring of G. Moreover, we can suppose that $s_2 > 1$, as P_∞ would be S-colorable if $s_2 = 1$. There exists a tree decomposition of G, (W, F, X) where
$W = \{u | u \in X_1\} \cup \{N(u) | u \in X_1\} \cup (V(G) \setminus X_1)$;
$F = \{(u, N(u)) | u \in X_1\} \cup \{((N(u), V(G) \setminus X_1) | u \in X_1\}$;
$X_u = \{u\}$; $X_{N(u)} = N(u)$; $X_{V(G)\setminus X_1} = V(G) \setminus X_1$.

Property 1 follows from the fact that $(V(G)\setminus X_1) \cup X_1 = V(G)$. Property 2 is satisfied, as in an edge uv in $E(G)$, the vertices are either in $V(G) \setminus X_1$ or in N_u for some u. For Property 3, suppose u and v are in X_1 and $u \neq v$. The different inequalities are true: $\{u\} \cap \{v\} = \emptyset$, $\{u\} \cap N(v) = \emptyset$, $\{u\} \cap (V(G) \setminus X_1) = \emptyset$ and $N(v) \cap N(u) \subseteq (V(G) \setminus X_1)$. For the last case, only $V(G) \setminus X_1$ lie on the path between $N(v)$ and $N(u)$. Therefore (W, F, X) is a tree decomposition of G.

As the vertices in X_1 can be colored 1, there have degree at most $|S| - 1$. Hence, for each u we have $|N(u)| \leq |S|$. Moreover, if a graph admits an $(s_2, \ldots, s_{|S|})$-coloring with $s_2 > 1$, then it cannot contain vertices of degree at least $|S| - 1$. Thus, using the Moore bound on the graph $G \setminus X_1$, $|V(G) \setminus X_1| < n(|S| - 1, p_{max}(S))$. Therefore, the treewidth of G is bounded. □

Proposition 2.13 ([5])**.** *Every S-colorable graph, for $S = (s_1, \ldots, s_{\lfloor 3i/2 \rfloor})$ and $s_1 = s_{\lfloor 3i/2 \rfloor} = i$, has treewidth as most $2i$.*

Corollary 2.14. *Every S-colorable graph, for $S = (s_1, \ldots, s_{\lfloor 3i/2 \rfloor})$ and $s_1 \geq i$, has treewidth as most $2i$.*

Proof. As every S-colorable graph, for $S = (s_1, \ldots, s_{\lfloor 3i/2 \rfloor})$ and $s_1 \geq i$, is S'-colorable, for $S' = (s'_1, \ldots, s'_{\lfloor 3i/2 \rfloor})$ and $s'_1 = s'_{\lfloor 3i/2 \rfloor} = i$, these graphs have treewidth at most $2i$. □

In the remainder of the paper, the elements of a sequence S' are denoted $s'_1, \ldots, s'_{|S'|}$.

Proposition 2.15 ([1])**.** *Let S be a non decreasing list of integers. The problem S-COL can be solved in polynomial time for graph of bounded treewidth.*

Corollary 2.16. *The problem S-COL can be solved in polynomial time for tree.*

Corollary 2.17. *The problem S-COL can be solved in polynomial time for $S = (s_1, \ldots, s_{\lfloor 3i/2 \rfloor})$ and $s_1 \geq i$.*

Proposition 2.18. *Let k be a positive integer and S be a non decreasing list of integers. If $\sum_{i=1}^{|S|} \frac{1}{s_i+1} < 1$, then S-COL can be solved in polynomial time.*

Moreover for $s_1 = 1$, *if* $\sum_{i=1}^{|S|} \frac{1}{2\lceil (s_i-1)/2 \rceil + 2} < 1$, *then* S*-COL can be solved in polynomial time.*

Proof. Using Propositions 2.10 and 2.11, we obtain that $p_{max}(S)$ is finite. Hence, using Proposition 2.12, every S-colorable graph has bounded treewidth. By Proposition 2.15, S-COL can be solved in polynomial time. □

3 NP-hard instances of S-COL

The following result establishes a dichotomy between NP-hard problems and polynomial time problems for $|S| = 3$.

Theorem 3.1 ([3])**.** *Let* $k \geq 1$ *be an integer. The problems* $(1, 1, k)$*-COL and* $(1, 2, 2)$*-COL are both NP-hard. Except these problems,* S*-COL is polynomial time solvable for* $|S| = 3$.

3.1 NP-hard instances of S-COL, for $s_1 \geq 2$

A. Sharp [4] has determined NP-hard instances of S-COL, for $s_1 = s_k = i \geq 2$. The next Theorem recall her result.

Theorem 3.2 ([4])**.** *The problem* S*-COL is NP-hard for* $s_1 = s_{|S|} = i$, $i > 1$ *and* $|S| \geq \lceil 3i/2 \rceil$.

In the following proposition, we establish the NP-hardness of the following problem:

Proposition 3.3. *The problem* S*-COL is NP-hard for* $i > 1$, $s_1 = s_{i+1} = i$ *and* $|S| \geq \lceil 3i/2 \rceil$.

Proof. Let N_j denotes the number of times the integer j appears in the list S and let $N_i^- = N_i - i + 1$.
The proof is by reduction from N_i-COL (as $N_i \geq 3$, N_i-COL is NP-hard).

> k**-COL**
> **Instance**: A graph G without leaves and an integer k.
> **Question**: Is there a partition of G into k independent sets.

Let G be a graph without leaves. Let $m = \prod_{j \in \mathbb{N} || N_j | \geq 1} (j+1)$. Let $L_{i,S}$ be the following graph: Take the path of length m with vertices $u_{0,0}, \ldots, u_{m-1,0}$. By the rank of a vertex in the path, we mean the first index of the vertex. By an l-insertion at rank j, we mean: adding a clique of l vertices $u_{j,j_{max}+1}, \ldots, u_{j,j_{max}+l}$, with $j_{max} = max\{k | u_{j,k} \in V(L_{i,S})\}$ and connect each vertex of the clique to every vertex of rank $j-1$, j or $j+1$. First, do an N_i^--insertion at rank j, for each j such that $j \equiv \lceil i/2 \rceil \pmod{i+1}$. Second, for every integer $l > i$, do an N_l-insertion at rank j, for each j such that $j > 0$ and $j \equiv 0 \pmod{l+1}$. The constructed graph is the graph $L'_{i,S}$ Third, take two copies of

$L'_{i,S}$ and merge the vertex $u_{0,0}$ in the first copy with the vertex $u_{0,0}$ in the second copy. Finally, do an $(|S| - N_i)$-insertion at rank 0 on this vertex $u_{0,0}$. The constructed graph is the graph $L_{i,s}$. The graph $J_{i,s}$ is the graph constructed as $L_{i,S}$ where one of the vertex $u_{m-1,0}$ from one copy of $L'_{i,S}$ is removed. Figure 1 illustrates a constructed $L_{i,S}$, for $i = 2$ and $S = (2, 2, 2, 2, 2, 3, 3)$. By connectors of $L_{i,s}$ or $J_{i,S}$ we denote the vertices $u_{m-1,0}$ of the two copies of $L'_{i,S}$ ($u_{m-2,0}$, if this vertex does not exist in $J_{i,S}$). Let $R_{i,S}$ be the graph from Figure 2. Observe that, depending the parity of i, the graph $R_{i,S}$ is different. By connectors of $R_{i,S}$, we denote the vertices c_1, c_2 and c_3 from Figure 2. By connecting, two copies of $R_{i,S}$ by $L_{i,S}$ (or $J_{i,S}$, respectively) we mean: taking two connectors of $L_{i,S}$ (of $J_{i,S}$, respectively), connecting one connector to a connector of the first copy of $R_{i,S}$ and connecting the other connector to a connector of the second copy of $R_{i,S}$, as in Figure 3. The used connectors are not consider connectors of $R_{i,S}$ anymore. We will construct a graph G', using the graph $L_{i,S}$, $J_{i,S}$ and $R_{i,S}$. For each vertex $v \in V(G)$ of degree d, we associate d copies of the graph $R_{i,S}$, denoted by $R^j_{i,S}$, $1 \geq j \geq d$. For each $0 < j < d$, we connect $R^j_{i,S}$ to $R^{j+1}_{i,S}$ by $L_{i,S}$, then we connect $R^1_{i,S}$ to $R^d_{i,S}$ by $L_{i,S}$. The obtained graph is G'_v and contains d connectors (each one in a different copy of $R_{i,S}$. For each edge $uv \in E(G)$, we connect G'_u to G'_v by $J_{i,S}$. The obtained graph is G'. By initial vertices of G', we denote the connectors of the graph G'_v, for each $v \in V(G)$. In order to prove that the construction is a reduction from $N_i - SAT$ to S-COL, we prove the following facts:

Fact 1: In every copy of $R_{i,S}$, the vertices c_1, c_2 and c_3 have the same color and this color is some i.

Remark that the diameter of $R_{i,S}$ is $i + 1$ and $|V(R_{i,S})| = |S| + 2$. Moreover, there exist only three pairs of vertices a distance $i+1$ and these pairs of vertices are composed of vertices in $\{c_1, c_2, c_3\}$. Thus, except c_1, c_2 and c_3 each vertex should be colored differently and there are only one color remaining, this color should be given to $\{c_1, c_2, c_3\}$ and this color should be a color i.

Fact 2: For every S-coloring of $L_{i,S}$ or $J_{i,S}$ in G', there is at most $2m/(s_j + 1)$ vertices colored s_j. Suppose the contrary. It means that there exist $2m/(s_j + 1) + 1$ vertices colored s_j in $L_{i,S}$ or $J_{i,S}$. In every case, a vertex colored s_j would be at distance less than $i + 1$ from a vertex colored s_j in a copy of $R_{i,S}$ ($R_{i,S}$ always contains a vertex colored s_j). Thus, we have a contradiction.

Fact 3: In every coloring of G' two connectors of different copies of $R_{i,s}$ have the same color if they are connected by $L_{i,S}$ and a different color if they are connected by $J_{i,S}$. Let s_k be the color of a connector of $R_{i,s}$. Using Fact 2, every color s_j appears at most $2m/(s_j + 1)$ times in $L_{i,S}$. $L_{i,S}$ contains less than $\prod_{x=1}^{|S|} 2m/(s_x + 1)$ vertices. A consequence is that for every l, there exists a vertex of rank l colored i. Therefore, using the fact that between two vertices colored s_k, there are vertices added by N_i^--insertion, we obtain that in every S-coloring of G', two connectors of different copies of $R_{i,s}$ have the same color. The argument is similar for $J_{i,S}$.

Fact 4: There exists an S-coloring of $L_{i,S}$ and $J_{i,S}$. First, color the vertices

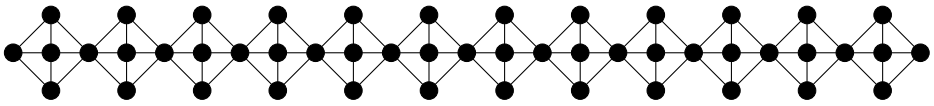

Figure 1: The graph $L_{2,(2,2,2,2,2,3,3)}$.

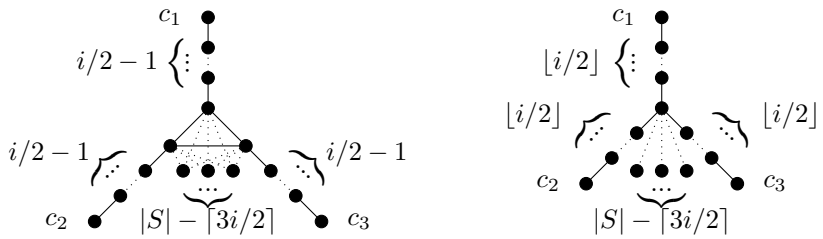

Figure 2: The graph $R_{i,S}$, for i even (on the left) and i odd (on the right).

$u_{0,k}$, for some k, alternating $i+1$ colors i. Second, for each N_l-insertion did in the construction, color the added vertices by the corresponding color l.
Fact 5: For any vertex v, in every S-coloring of D_v the original vertices have the same color i. Using Fact 1 and Fact 3, this fact is trivial.
Fact 6: In any S-coloring of G', any pair of colors in N_i can be given to original vertices connected by $J_{i,S}$. This fact is not trivial, this is a consequence of the following property: in every D_v, any color can be given to the vertices adjacent to a connector as Figure 3 illustrates it.
Suppose that G admits an N_i-coloring. We can prove that G' admits an S-coloring. We give the colors from G to original vertices of G'. To the color number j corresponds the color s_j of S. Using Fact 3 and Fact 6, the coloring can be extended to an S-coloring of G'. Suppose that G' admits an S-coloring, using Fact 3 and Fact 5, we can deduce that every original vertex is colored some color i. We give the colors from original vertices of G' to vertices of G. To the color $s_j = i$, corresponds the color number j.

□

3.2 From S-COL to S'-COL, for $|S'| = |S| + 1$

Proposition 3.4. *Let S be a non decreasing list of integers with $s_2 = 1$. If S-COL is NP-hard, then S'-COL is NP-hard for $|S'| = |S| + 1$ and $s'_i = s_i$ for every i, $1 \leq i \leq |S|$.*

Proof. Suppose S-COL is NP-hard and $s'_{|S|+1} = \ell$. Let N_ℓ be the number of integers ℓ in S. From a graph G, we construct a new graph G' obtained from G, by connecting every vertex v in $V(G)$ to a path of $|S|-1$ vertices with end vertex m_v. Adding one vertex m_f and connecting it to any end vertex of a path m_v. Taking a complete graph $K_{|S|+1}$ and making the join between the set

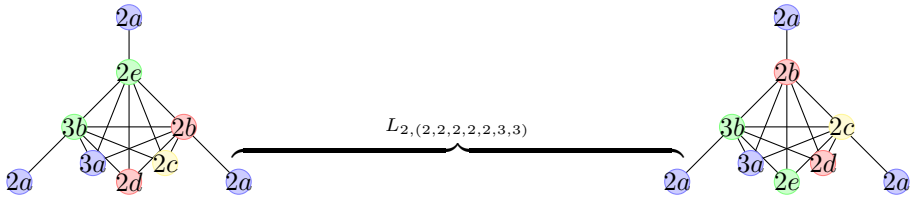

Figure 3: A S-coloring of two copies of $R_{2,S}$ connected by $L_{2,S}$, for $S = (2, 2, 2, 2, 2, 3, 3)$.

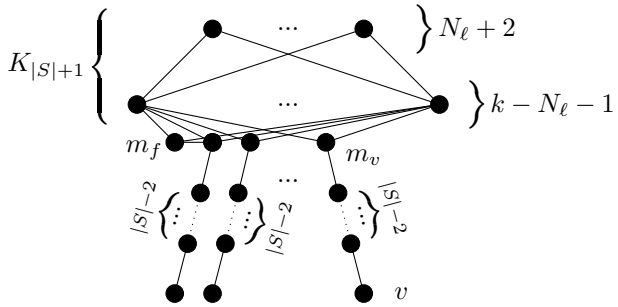

Figure 4: Illustration of the reduction for $s_2 = 1$.

$\{m_v | v \in V(G)\} \cup m_f$ and a set of $|S| - N_\ell - 1$ vertices in $K_{|S|+1}$. The graph obtained is the graph G'. By original vertices of G', we denote the vertices that come from G.

Suppose G admits an S-coloring. We gives the S-coloring of G to original vertices of G'. Afterwards, for every vertex v we alternate colors $s_1 = 1a$ and $s_2 = 1b$ for every path until m_v. We give the color $1a$ or $1b$ to m_f (depending the color of the vertex connected to m_f). We have $N_\ell + 2$ vertices in $K_{|S|+1}$ not connected to m_v. Colors $1a$, $1b$ and every color ℓ in S are given to these vertices. The remaining colors are given to the vertices not colored in $K_{|S|+1}$.

Suppose that G' admits an S'-coloring. In $K_{|S|+1}$, every color should be given. As the two colors given to m_v and m_f should be used on the $N_\ell + 2$ vertices in $K_{|S|+1}$ not connected to m_v, at most N_ℓ vertices could be colored ℓ in the $N_\ell + 2$ vertices in $K_{|S|+1}$ not connected to m_v. Hence, one vertex should be colored ℓ in the remaining vertices of $K_{|S|+1}$. Thus, one color ℓ cannot be used in the remaining graph. The coloring used on the initial vertices of G' gives us an S-coloring of G. □

Proposition 3.5. *Let S be a non decreasing list of integers. If S-COL is NP-hard, then S'-COL is NP-hard, for every S' such that $|S'| = |S| + 1$, $s'_1 = 1$ and $s'_{i+1} = 2s_i$ or $2s_i + 1$, for every i, $1 \leq i \leq |S|$.*

Proof. Suppose that S-COL is NP-hard. There exists a reduction from S-COL to S'-COL completely similar to the reduction introduced by Goddard et al. [2] from $(1,1,2)$-COL to $(1,2,3,4)$-COL. Let G be a connected graph. We build a new graph G' where we replace each edge uv by $|S|+1$ parallel edges and then subdivide each edge. The vertices u and v are called initial vertices.
Let G be a graph which admits an S-coloring and let $X_i \subseteq V(G)$ be an i-packing. Every vertices u and v in X_i are such that $d(u,v) > i$. Let $X'_i \subseteq V(G')$ be the set of initial vertices in X_i. Observe that every vertices u and v in X'_i are such that $d_{G'}(u,v) > 2i+1$. To conclude, X'_i is an $(2i+1)$-packing (therefore an $2i$-packing of G'). Let X'_1 be the set of vertices created by subdividing each edge. This set is an independent set and can be colored 1. Hence, there exists $|S|+1$ sets, $X'_1, X'_{s_1}, \ldots, X'_{s_{|S|}}$ that form a partition of G'. Thus, G' admits a S'-coloring.
Suppose that G' admits an S'-coloring. Suppose that there exists an initial vertex u colored 1. Since u has $|S|+1$ neighbors, then two of these neighbors have the same color s'_i. The fact that $s'_i > 1$ gives us that this coloring cannot be extended in an S'-coloring. Hence, no initial vertices are colored 1. Using the S'-coloring of the initial vertices in G' we can create an S-coloring of G by giving the color s_{i+1} to the vertices colored s'_i in G'. □

Proposition 3.6. *The problem (s_1,s_2,s_3,s_4)-COL is NP-hard for $s_1 = 1$, $s_2 \leq 3$, $s_3 \leq 5$ and $s_4 \leq 5$, for $s_1 = 1$, $s_2 \leq 3$, $s_3 \leq 3$ and $s_4 \geq 3$ and for $s_1 = 2$, $s_2 = 2$, $s_3 = 2$ and $s_4 \geq 2$. The problem (s_1,s_2,s_3,s_4)-COL is polynomial time solvable for $s_1 \geq 3$, for $s_1 = 1$ and $s_2 > 3$, for $s_1 = 1$, $2 \leq s_2 \leq 3$ and $s_3 > 7$, for $s_1 = 1$, $2 \leq s_2 \leq 3$, $4 \leq s_3 \leq 5$ and $s_4 > 11$ and for $s_1 = 1$, $2 \leq s_2 \leq 3$, $6 \leq s_3 \leq 7$ and $s_4 > 7$.*

Proof. Using Proposition 3.4, we obtain that for every sequence S beginning with 1, 1 and $k \geq 1$, S-COL is NP-hard. Moreover, using Proposition 3.5 from $(1,2,2)$-COL and $(1,1,k)$-COL, $k \geq 1$, we obtain (s_1,s_2,s_3,s_4)-COL is NP-hard, for $s_1 = 1$, $s_2 \leq 3$, $s_3 \leq 5$ and $s_4 \leq 5$ and for $s_1 = 1$, $s_2 \leq 3$, $s_3 \leq 3$ and $s_4 \geq 3$. Finally, using Proposition 3.3 we obtain (s_1,s_2,s_3,s_4)-COL is NP-hard, for $s_1 = 2$, $s_2 = 2$, $s_3 = 2$ and $s_4 \geq 2$.
Moreover, by Proposition 2.18, we can see that in the different cases proved to be polynomial, P_∞ does not admit an S-coloring. Hence, these cases are polynomial. For $s_1 = s_4 = 3$, Corollary 2.17 is used. □

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