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On the Complexity of the S -coloring problem

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Abstract

This work establishes the complexity class of several instances of the S -coloring problem: For a graph G , a positive integer k and a non decreasing list of integers $S = (s_1, \dots, s_k)$, G admits a S -coloring, if its vertices can be partitioned into sets X_{s_i} , $i = 1, \dots, k$, where each X_{s_i} being an s_i -packing (a set of vertices at pairwise distance greater than s_i). For a unfixed size of list, the complexity of the S -coloring problem is determined for several instances of the problem.

Keywords: NP-hard problem, distance, Packing chromatic number, d -distance coloring.

1 Introduction

We consider only undirected connected graphs in this paper. Given a graph $G = (V, E)$, an i -packing is a set $X_i \subseteq V(G)$ such that for any distinct pair $u, v \in X_i$, $d_G(u, v) > i$, where $d_G(u, v)$ denotes the usual shortest path distance between u and v . We will use X_i to refer to an i -packing in a graph G . For a non decreasing sequence of positive integers $S = (s_i, i \in \mathbb{N}^*)$, an S - k -coloring of G is a partition of $V(G)$ into sets X_{s_1}, \dots, X_{s_k} , where each X_{s_i} being an s_i -packing. For a non decreasing list $S' = (s_1, \dots, s_k)$ of k integers, an S' - $(packing)$ -coloring of G is a S - k -coloring of G for a sequence S which begins with a list S' . A graph G is S -colorable if there exists a S -coloring of G . By $|S|$, we denote the size of the list S , and by s_i we denote the i th element of the list. Then we define the following decision problem:

S -COL

Instance : A graph G

Question: Does G have an S -coloring ?

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The problem $(1, \dots, 1)$ -COL corresponds to the k -coloring problem (where k is the number of 1) which is known to be NP-complete for $k \geq 3$. The S -coloring problem generalizes coloring problems with distance constraints like the *packing chromatic number* or the *distance chromatic number* of a graph. The packing chromatic number [2] of G is the least integer k such that the vertex set of G can be partitioned into sets X_i , $i = 1, \dots, k$, where each X_i being an i -packing. Hence, the least integer k such that G has a $(1, 2, \dots, k)$ -coloring. The d -distance chromatic number [4] of G is the least integer k such that the vertex set of G can be partitioned into sets X_d^i , $i = 1, \dots, k$, where each X_d^i being a d -packing. Initially, the S -coloring has been introduced by Goddard et al. [2] and Fiala et al. [1]. The S -coloring problem, with $|S| = 3$ has been studied by Goddard et al. [2] in order to determine the complexity of the packing chromatic number when $k = 4$. Moreover, Goddard and Xu [3] have proven that for $|S| = 3$, S -COL is NP-hard, if $s_1 = s_2 = 1$ or if $s_1 = 1$ and $s_2 = s_3 = 2$ and polynomial otherwise.

In order to represent an S -coloring, we will denote colors of vertices in different i -packings by i followed by a letter of the alphabet ($2a$ and $2b$ for example).

Remark 1.1. Notice that the $(1, 1)$ -colorable graphs correspond to bipartite graphs. If $s_2 > 1$, only trees of diameter 2 are (s_1, s_2) -colorable. Hence, (s_1, s_2) -COL is polynomial time solvable.

In the second section, properties on the S -coloring of paths are used to determine polynomial instances of S -COL. In the third section, various NP-hard instances of S -COL are determined.

2 S -coloring of path and polynomial instances of S -COL

2.1 S -coloring of infinite graphs

Proposition 2.1. Let S be a non decreasing list of integers, with $s_1 > 1$ and G be a graph. If G contains a vertex of degree $|S|$, then G is not S -colorable.

Proof. Let v be a vertex of degree $|S|$. As v and the neighbors of v are at pairwise distance at most 2, every vertex must have a different color and we only have $|S|$ colors. \square

Definition 2.1. Let k and d be two positive integers. By $n(k, d) = 1 + k \sum_{i=0}^{d-1} (k - 1)^i$, we denote the Moore bound. The Moore bound corresponds to the maximum number of vertices for a graph of diameter d and maximal degree k .

Proposition 2.2. Let S be a non decreasing list of integers, with $s_1 > 1$. If an infinite graph is S -colorable, then P_∞ is S -colorable.

Proof. Notice that a graph of diameter d has the graph P_d as subgraph. Hence, if an infinite graph with a unbounded diameter is S -colorable, then P_∞ is S -colorable. By contradiction, we suppose there exists an infinite S -colorable graph G with a fixed diameter d , $d > 0$. By Proposition 2.1, the maximum degree of G is $|S| - 1$. Therefore, using the Moore bound, G has order at most $n(|S| - 1, d)$. \square

Definition 2.2. Let S be a non decreasing list of integers. Let $t_{max}(S)$ be the order of the largest graph which admits an S -coloring. If $t_{max}(S)$ is not bounded, then we define $t_{max}(S)$ as ∞ .

Claim 2.3. Let S be a non decreasing list of integers. If $t_{max}(S)$ is bounded, then there exists an S -colorable tree of order $t_{max}(S)$.

Proposition 2.4. Let S be a non decreasing list of integers. If $s_1 \geq |S|$ then $t_{max}(S) = |S|$.

Proof. A graph of order $|S| + 1$ has diameter at most $|S|$. Hence, every vertex should be colored differently. \square

2.2 S -coloring of the path

For a vertex $x \in V(G)$, the ball of radius n centered at x is the set $B_n(x) = \{v \in V(G) | d_G(x, v) \leq n\}$. The density of a set of vertices $X \subseteq V(G)$ is $d_G(X) = \limsup_{l \rightarrow \infty} \max_{x \in V} \frac{|X \cap B_l(x)|}{|B_l(x)|}$. Suppose P_∞ admits an S -coloring and let v be a vertex of P_∞ colored c . Let $N_c(v)$ be the set of vertices (no more than two vertices) which have the color c such that in the smallest path between each of these vertices and v , no vertex is colored c .

Proposition 2.5. Let S be a non decreasing list of integers. For every S -coloring of P_∞ there exists another S -coloring such that for every vertex v , if v is colored s_i , for some i , then every vertex in $N_{s_i}(v)$ is at distance at most $2s_i - 1$ from v .

Proof. Suppose that two vertices u and v colored s_i are such that $d(u, v) \geq 2s_i$ with no vertex colored s_i in the path between u and v . Let $l = \lfloor d(u, v)/s_i \rfloor$ and let $u_1, \dots, u_{d(u, v)-1}$ be the vertices in the path between u and v . By adding a vertex between u and v , we mean remove the edge uv , add a vertex w and connect it to u and v . We add one vertex colored s_i between the vertices $u_{l's_i}$ and $u_{(l'+1)s_i}$, for each l' such that $0 < l' < l$. We obtain an S -coloring such that for every vertex v colored s_i , every vertex in $N_{s_i}(v)$ is at distance at most $2s_i - 1$ from v . \square

Corollary 2.6. Let S be a non decreasing list of integers with $s_1 = 1$. For every S -coloring of P_∞ there exists another S -coloring such that for every vertex v colored 1, the vertices in $N_1(v)$ are at distance 2.

Definition 2.3. Let G be a graph. The maximum density of an i -packing in G is denoted $d_G(X_i)$.

Claim 2.7. Let S be a non decreasing list of integers. If $\sum_{i=1}^{|S|} (d_G(X_{s_i}) < 1\}$, then G is not S -colorable.

Claim 2.8. If $X_i \subseteq V(P_\infty)$ is an i -packing of P_∞ , then $d(X_i) \leq \frac{1}{i+1}$.

Definition 2.4. Let S be a non decreasing list of integers. Let $p_{max}(S)$ be the order of the largest S -colorable path. If P_∞ is S -colorable, then we define $p_{max}(S)$ as ∞ .

Remark 2.9. Let S be a non decreasing list of integers. First, remark that $p_{max}(S) \leq t_{max}(S)$. Second, if $p_{max}(S)$ is bounded and $s_1 > 2$, then using the Moore bound, we have $t_{max}(S) \leq n(|S| - 1, p_{max}(S))$.

Proposition 2.10. Let S be a non decreasing list of integers. If $\sum_{i=1}^{|S|} \frac{1}{s_i+1} < 1$, then $p_{max}(S) < \infty$.

Proof. Claim 2.8 allows to conclude that $\sum_{i=1}^{|S|} d(X_{s_i}) \leq \sum_{i=1}^{|S|} \frac{1}{s_i+1} < 1$. Thus, using claim 2.7 P_∞ does not admit an S -coloring. \square

Proposition 2.11. Let S be a non decreasing list of integers, with $s_1 = 1$. If $\sum_{i=1}^{|S|} \frac{1}{2^{\lceil (s_i-1)/2 \rceil + 2}} < 1$, then $p_{max}(S) < \infty$.

Proof. Suppose that $s_1 = 1$, Corollary 2.6 gives us that there exists an S -coloring where X_1 , the set of vertices colored by 1, is one of the two independent sets that form a bipartition of P_∞ . A consequence is that any i -packing, with i even, is an $(i+1)$ -packing too. We obtain that if $\sum_{i=1}^{|S|} \frac{1}{2^{\lceil (s_i-1)/2 \rceil + 2}} < 1$, then $p_{max}(S) < \infty$ (the formula corresponds to add one to even numbers). \square

2.3 Polynomial instances of S -COL

Definition 2.5. A Tree decomposition of a graph $G = (V, E)$ is a triple (T, F, X) consisting of an undirected tree (T, F) and a map $X : T \rightarrow 2^{|V|}$ associating to each vertex $i \in T$, a subset $X_i \subseteq V$ such that:

1. $V = \bigcup_{i \in T} X_i$;
2. For all edges $uv \in E$, there exists $i \in T$ such that u and v are in X_i ;
3. If j lies on the path between i and k in (T, F) then $X_i \cap X_k \subseteq X_j$.

The width of (T, F, X) is $\max_{i \in T} (|X_i| - 1)$. The treewidth of G is the smallest width of a tree decomposition of G .

For a given list S such that $p_{max}(S) < \infty$, we aim to bound the treewidth of the S -colorable graphs, isolating vertices of degree at least k .

Proposition 2.12. *For a given S -colorable graph G with $p_{max}(S) < \infty$, the treewidth of G is bounded.*

Proof. If $s_1 > 1$ and $p_{max}(S) < \infty$, then by Proposition 2.2 the size of G is bounded. Thus, the treewidth of G is bounded. Suppose that $s_1 = 1$ and let X_1 be the set of vertices colored by 1 for a fixed S -coloring of G . Moreover, we can suppose that $s_2 > 1$, as P_∞ would be S -colorable if $s_2 = 1$. There exists a tree decomposition of G , (W, F, X) where

$$\begin{aligned} W &= \{u|u \in X_1\} \cup \{N(u)|u \in X_1\} \cup (V(G) \setminus X_1); \\ F &= \{(u, N(u))|u \in X_1\} \cup \{(N(u), V(G) \setminus X_1)|u \in X_1\}; \\ X_u &= \{u\}; X_{N(u)} = N(u); X_{V(G) \setminus X_1} = V(G) \setminus X_1. \end{aligned}$$

Property 1 follows from the fact that $(V(G) \setminus X_1) \cup X_1 = V(G)$. Property 2 is satisfied, as in an edge uv in $E(G)$, the vertices are either in $V(G) \setminus X_1$ or in N_u for some u . For Property 3, suppose u and v are in X_1 and $u \neq v$. The different inequalities are true: $\{u\} \cap \{v\} = \emptyset$, $\{u\} \cap N(v) = \emptyset$, $\{u\} \cap (V(G) \setminus X_1) = \emptyset$ and $N(v) \cap N(u) \subseteq (V(G) \setminus X_1)$. For the last case, only $V(G) \setminus X_1$ lie on the path between $N(v)$ and $N(u)$. Therefore (W, F, X) is a tree decomposition of G .

As the vertices in X_1 can be colored 1, there have degree at most $|S| - 1$. Hence, for each u we have $|N(u)| \leq |S|$. Moreover, if a graph admits an $(s_2, \dots, s_{|S|})$ -coloring with $s_2 > 1$, then it cannot contain vertices of degree at least $|S| - 1$. Thus, using the Moore bound on the graph $G \setminus X_1$, $|V(G) \setminus X_1| < n(|S| - 1, p_{max}(S))$. Therefore, the treewidth of G is bounded. \square

Proposition 2.13 ([5]). *Every S -colorable graph, for $S = (s_1, \dots, s_{\lfloor 3i/2 \rfloor})$ and $s_1 = s_{\lfloor 3i/2 \rfloor} = i$, has treewidth at most $2i$.*

Corollary 2.14. *Every S -colorable graph, for $S = (s_1, \dots, s_{\lfloor 3i/2 \rfloor})$ and $s_1 \geq i$, has treewidth at most $2i$.*

Proof. As every S -colorable graph, for $S = (s_1, \dots, s_{\lfloor 3i/2 \rfloor})$ and $s_1 \geq i$, is S' -colorable, for $S' = (s'_1, \dots, s'_{\lfloor 3i/2 \rfloor})$ and $s'_1 = s'_{\lfloor 3i/2 \rfloor} = i$, these graphs have treewidth at most $2i$. \square

In the remainder of the paper, the elements of a sequence S' are denoted $s'_1, \dots, s'_{|S'|}$.

Proposition 2.15 ([1]). *Let S be a non decreasing list of integers. The problem S -COL can be solved in polynomial time for graph of bounded treewidth.*

Corollary 2.16. *The problem S -COL can be solved in polynomial time for tree.*

Corollary 2.17. *The problem S -COL can be solved in polynomial time for $S = (s_1, \dots, s_{\lfloor 3i/2 \rfloor})$ and $s_1 \geq i$.*

Proposition 2.18. *Let k be a positive integer and S be a non decreasing list of integers. If $\sum_{i=1}^{|S|} \frac{1}{s_i+1} < 1$, then S -COL can be solved in polynomial time.*

Moreover for $s_1 = 1$, if $\sum_{i=1}^{|S|} \frac{1}{2^{\lceil (s_i-1)/2 \rceil + 2}} < 1$, then S -COL can be solved in polynomial time.

Proof. Using Propositions 2.10 and 2.11, we obtain that $p_{max}(S)$ is finite. Hence, using Proposition 2.12, every S -colorable graph has bounded treewidth. By Proposition 2.15, S -COL can be solved in polynomial time. \square

3 NP-hard instances of S -COL

The following result establishes a dichotomy between NP-hard problems and polynomial time problems for $|S| = 3$.

Theorem 3.1 ([3]). *Let $k \geq 1$ be an integer. The problems $(1, 1, k)$ -COL and $(1, 2, 2)$ -COL are both NP-hard. Except these problems, S -COL is polynomial time solvable for $|S| = 3$.*

3.1 NP-hard instances of S -COL, for $s_1 \geq 2$

A. Sharp [4] has determined NP-hard instances of S -COL, for $s_1 = s_k = i \geq 2$. The next Theorem recall her result.

Theorem 3.2 ([4]). *The problem S -COL is NP-hard for $s_1 = s_{|S|} = i$, $i > 1$ and $|S| \geq \lceil 3i/2 \rceil$.*

In the following proposition, we establish the NP-hardness of the following problem:

Proposition 3.3. *The problem S -COL is NP-hard for $i > 1$, $s_1 = s_{i+1} = i$ and $|S| \geq \lceil 3i/2 \rceil$.*

Proof. Let N_j denotes the number of times the integer j appears in the list S and let $N_i^- = N_i - i + 1$.

The proof is by reduction from N_i -COL (as $N_i \geq 3$, N_i -COL is NP-hard).

k -COL

Instance: A graph G without leaves and an integer k .

Question: Is there a partition of G into k independent sets.

Let G be a graph without leaves. Let $m = \prod_{j \in \mathbb{N} | N_j \geq 1} (j + 1)$. Let $L_{i,S}$ be

the following graph: Take the path of length m with vertices $u_{0,0}, \dots, u_{m-1,0}$. By the rank of a vertex in the path, we mean the first index of the vertex. By an l -insertion at rank j , we mean: adding a clique of l vertices $u_{j,j_{max}+1}, \dots, u_{j,j_{max}+l}$, with $j_{max} = \max\{k | u_{j,k} \in V(L_{i,S})\}$ and connect each vertex of the clique to every vertex of rank $j-1$, j or $j+1$. First, do an N_i^- -insertion at rank j , for each j such that $j \equiv \lceil i/2 \rceil \pmod{i+1}$. Second, for every integer $l > i$, do an N_l -insertion at rank j , for each j such that $j > 0$ and $j \equiv 0 \pmod{l+1}$. The constructed graph is the graph $L'_{i,S}$. Third, take two copies of

$L'_{i,S}$ and merge the vertex $u_{0,0}$ in the first copy with the vertex $u_{0,0}$ in the second copy. Finally, do an $(|S| - N_i)$ -insertion at rank 0 on this vertex $u_{0,0}$. The constructed graph is the graph $L_{i,S}$. The graph $J_{i,S}$ is the graph constructed as $L_{i,S}$ where one of the vertex $u_{m-1,0}$ from one copy of $L'_{i,S}$ is removed. Figure 1 illustrates a constructed $L_{i,S}$, for $i = 2$ and $S = (2, 2, 2, 2, 2, 3, 3)$. By connectors of $L_{i,S}$ or $J_{i,S}$ we denote the vertices $u_{m-1,0}$ of the two copies of $L'_{i,S}$ ($u_{m-2,0}$, if this vertex does not exist in $J_{i,S}$). Let $R_{i,S}$ be the graph from Figure 2. Observe that, depending the parity of i , the graph $R_{i,S}$ is different. By connectors of $R_{i,S}$, we denote the vertices c_1 , c_2 and c_3 from Figure 2. By connecting, two copies of $R_{i,S}$ by $L_{i,S}$ (or $J_{i,S}$, respectively) we mean: taking two connectors of $L_{i,S}$ (of $J_{i,S}$, respectively), connecting one connector to a connector of the first copy of $R_{i,S}$ and connecting the other connector to a connector of the second copy of $R_{i,S}$, as in Figure 3. The used connectors are not consider connectors of $R_{i,S}$ anymore. We will construct a graph G' , using the graph $L_{i,S}$, $J_{i,S}$ and $R_{i,S}$. For each vertex $v \in V(G)$ of degree d , we associate d copies of the graph $R_{i,S}$, denoted by $R_{i,S}^j$, $1 \geq j \geq d$. For each $0 < j < d$, we connect $R_{i,S}^j$ to $R_{i,S}^{j+1}$ by $L_{i,S}$, then we connect $R_{i,S}^1$ to $R_{i,S}^d$ by $L_{i,S}$. The obtained graph is G'_v and contains d connectors (each one in a different copy of $R_{i,S}$. For each edge $uv \in E(G)$, we connect G'_u to G'_v by $J_{i,S}$. The obtained graph is G' . By initial vertices of G' , we denote the connectors of the graph G'_v , for each $v \in V(G)$. In order to prove that the construction is a reduction from $N_i - SAT$ to S -COL, we prove the following facts:

Fact 1: In every copy of $R_{i,S}$, the vertices c_1 , c_2 and c_3 have the same color and this color is some i .

Remark that the diameter of $R_{i,S}$ is $i + 1$ and $|V(R_{i,S})| = |S| + 2$. Moreover, there exist only three pairs of vertices a distance $i + 1$ and these pairs of vertices are composed of vertices in $\{c_1, c_2, c_3\}$. Thus, except c_1 , c_2 and c_3 each vertex should be colored differently and there are only one color remaining, this color should be given to $\{c_1, c_2, c_3\}$ and this color should be a color i .

Fact 2: For every S -coloring of $L_{i,S}$ or $J_{i,S}$ in G' , there is at most $2m/(s_j + 1)$ vertices colored s_j . Suppose the contrary. It means that there exist $2m/(s_j + 1) + 1$ vertices colored s_j in $L_{i,S}$ or $J_{i,S}$. In every case, a vertex colored s_j would be at distance less than $i + 1$ from a vertex colored s_j in a copy of $R_{i,S}$ ($R_{i,S}$ always contains a vertex colored s_j). Thus, we have a contradiction.

Fact 3: In every coloring of G' two connectors of different copies of $R_{i,S}$ have the same color if they are connected by $L_{i,S}$ and a different color if they are connected by $J_{i,S}$. Let s_k be the color of a connector of $R_{i,S}$. Using Fact 2, every color s_j appears at most $2m/(s_j + 1)$ times in $L_{i,S}$. $L_{i,S}$ contains less

than $\prod_{x=1}^{|S|} 2m/(s_x + 1)$ vertices. A consequence is that for every l , there exists a vertex of rank l colored i . Therefore, using the fact that between two vertices colored s_k , there are vertices added by N_i^- -insertion, we obtain that in every S -coloring of G' , two connectors of different copies of $R_{i,S}$ have the same color. The argument is similar for $J_{i,S}$.

Fact 4: There exists an S -coloring of $L_{i,S}$ and $J_{i,S}$. First, color the vertices

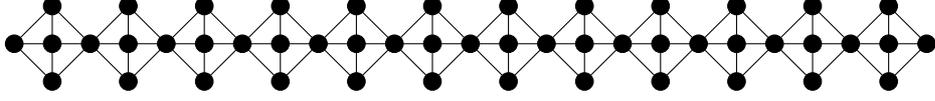


Figure 1: The graph $L_{2,(2,2,2,2,2,3,3)}$.

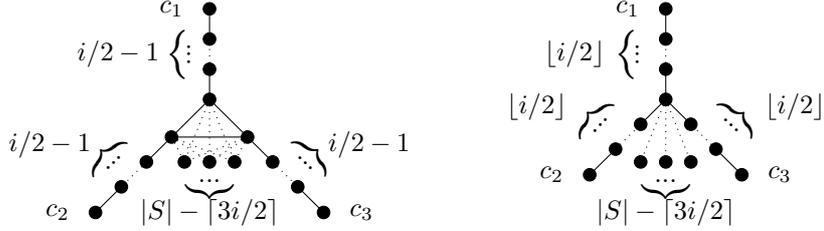


Figure 2: The graph $R_{i,S}$, for i even (on the left) and i odd (on the right).

$u_{0,k}$, for some k , alternating $i + 1$ colors i . Second, for each N_l -insertion did in the construction, color the added vertices by the corresponding color l .

Fact 5: For any vertex v , in every S -coloring of D_v the original vertices have the same color i . Using Fact 1 and Fact 3, this fact is trivial.

Fact 6: In any S -coloring of G' , any pair of colors in N_i can be given to original vertices connected by $J_{i,S}$. This fact is not trivial, this is a consequence of the following property: in every D_v , any color can be given to the vertices adjacent to a connector as Figure 3 illustrates it.

Suppose that G admits an N_i -coloring. We can prove that G' admits an S -coloring. We give the colors from G to original vertices of G' . To the color number j corresponds the color s_j of S . Using Fact 3 and Fact 6, the coloring can be extended to an S -coloring of G' . Suppose that G' admits an S -coloring, using Fact 3 and Fact 5, we can deduce that every original vertex is colored some color i . We give the colors from original vertices of G' to vertices of G . To the color $s_j = i$, corresponds the color number j .

□

3.2 From S -COL to S' -COL, for $|S'| = |S| + 1$

Proposition 3.4. *Let S be a non decreasing list of integers with $s_2 = 1$. If S -COL is NP-hard, then S' -COL is NP-hard for $|S'| = |S| + 1$ and $s'_i = s_i$ for every i , $1 \leq i \leq |S|$.*

Proof. Suppose S -COL is NP-hard and $s'_{|S|+1} = \ell$. Let N_ℓ be the number of integers ℓ in S . From a graph G , we construct a new graph G' obtained from G , by connecting every vertex v in $V(G)$ to a path of $|S| - 1$ vertices with end vertex m_v . Adding one vertex m_f and connecting it to any end vertex of a path m_v . Taking a complete graph $K_{|S|+1}$ and making the join between the set

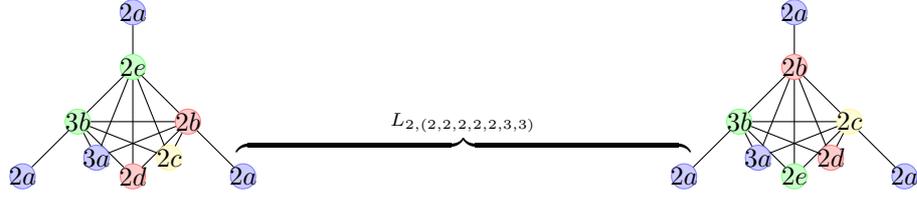


Figure 3: A S -coloring of two copies of $R_{2,S}$ connected by $L_{2,S}$, for $S = (2, 2, 2, 2, 2, 3, 3)$.

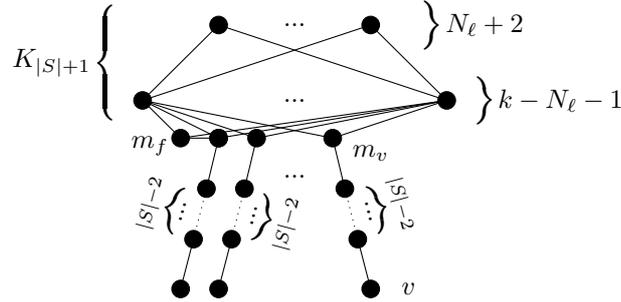


Figure 4: Illustration of the reduction for $s_2 = 1$.

$\{m_v | v \in V(G)\} \cup m_f$ and a set of $|S| - N_\ell - 1$ vertices in $K_{|S|+1}$. The graph obtained is the graph G' . By original vertices of G' , we denote the vertices that come from G .

Suppose G admits an S -coloring. We give the S -coloring of G to original vertices of G' . Afterwards, for every vertex v we alternate colors $s_1 = 1a$ and $s_2 = 1b$ for every path until m_v . We give the color $1a$ or $1b$ to m_f (depending on the color of the vertex connected to m_f). We have $N_\ell + 2$ vertices in $K_{|S|+1}$ not connected to m_v . Colors $1a, 1b$ and every color ℓ in S are given to these vertices. The remaining colors are given to the vertices not colored in $K_{|S|+1}$. Suppose that G' admits an S' -coloring. In $K_{|S|+1}$, every color should be given. As the two colors given to m_v and m_f should be used on the $N_\ell + 2$ vertices in $K_{|S|+1}$ not connected to m_v , at most N_ℓ vertices could be colored ℓ in the $N_\ell + 2$ vertices in $K_{|S|+1}$ not connected to m_v . Hence, one vertex should be colored ℓ in the remaining vertices of $K_{|S|+1}$. Thus, one color ℓ cannot be used in the remaining graph. The coloring used on the initial vertices of G' gives us an S -coloring of G . \square

Proposition 3.5. *Let S be a non decreasing list of integers. If S -COL is NP-hard, then S' -COL is NP-hard, for every S' such that $|S'| = |S| + 1$, $s'_1 = 1$ and $s'_{i+1} = 2s_i$ or $2s_i + 1$, for every i , $1 \leq i \leq |S|$.*

Proof. Suppose that S -COL is NP-hard. There exists a reduction from S -COL to S' -COL completely similar to the reduction introduced by Goddard et al. [2] from $(1, 1, 2)$ -COL to $(1, 2, 3, 4)$ -COL. Let G be a connected graph. We build a new graph G' where we replace each edge uv by $|S| + 1$ parallel edges and then subdivide each edge. The vertices u and v are called initial vertices.

Let G be a graph which admits an S -coloring and let $X_i \subseteq V(G)$ be an i -packing. Every vertices u and v in X_i are such that $d(u, v) > i$. Let $X'_i \subseteq V(G')$ be the set of initial vertices in X_i . Observe that every vertices u and v in X'_i are such that $d_{G'}(u, v) > 2i + 1$. To conclude, X'_i is an $(2i + 1)$ -packing (therefore an $2i$ -packing of G'). Let X'_1 be the set of vertices created by subdividing each edge. This set is an independent set and can be colored 1. Hence, there exists $|S| + 1$ sets, $X'_1, X'_{s_1}, \dots, X'_{s_{|S|}}$ that form a partition of G' . Thus, G' admits a S' -coloring.

Suppose that G' admits an S' -coloring. Suppose that there exists an initial vertex u colored 1. Since u has $|S| + 1$ neighbors, then two of these neighbors have the same color s'_i . The fact that $s'_i > 1$ gives us that this coloring cannot be extended in an S' -coloring. Hence, no initial vertices are colored 1. Using the S' -coloring of the initial vertices in G' we can create an S -coloring of G by giving the color s_{i+1} to the vertices colored s'_i in G' . \square

Proposition 3.6. *The problem (s_1, s_2, s_3, s_4) -COL is NP-hard for $s_1 = 1, s_2 \leq 3, s_3 \leq 5$ and $s_4 \leq 5$, for $s_1 = 1, s_2 \leq 3, s_3 \leq 3$ and $s_4 \geq 3$ and for $s_1 = 2, s_2 = 2, s_3 = 2$ and $s_4 \geq 2$. The problem (s_1, s_2, s_3, s_4) -COL is polynomial time solvable for $s_1 \geq 3$, for $s_1 = 1$ and $s_2 > 3$, for $s_1 = 1, 2 \leq s_2 \leq 3$ and $s_3 > 7$, for $s_1 = 1, 2 \leq s_2 \leq 3, 4 \leq s_3 \leq 5$ and $s_4 > 11$ and for $s_1 = 1, 2 \leq s_2 \leq 3, 6 \leq s_3 \leq 7$ and $s_4 > 7$.*

Proof. Using Proposition 3.4, we obtain that for every sequence S beginning with 1, 1 and $k \geq 1$, S -COL is NP-hard. Moreover, using Proposition 3.5 from $(1, 2, 2)$ -COL and $(1, 1, k)$ -COL, $k \geq 1$, we obtain (s_1, s_2, s_3, s_4) -COL is NP-hard, for $s_1 = 1, s_2 \leq 3, s_3 \leq 5$ and $s_4 \leq 5$ and for $s_1 = 1, s_2 \leq 3, s_3 \leq 3$ and $s_4 \geq 3$. Finally, using Proposition 3.3 we obtain (s_1, s_2, s_3, s_4) -COL is NP-hard, for $s_1 = 2, s_2 = 2, s_3 = 2$ and $s_4 \geq 2$.

Moreover, by Proposition 2.18, we can see that in the different cases proved to be polynomial, P_∞ does not admit an S -coloring. Hence, these cases are polynomial. For $s_1 = s_4 = 3$, Corollary 2.17 is used. \square

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