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Dichotomies properties on computational complexity of S -packing coloring problems

Nicolas Gastineau^{*1,2}

¹*LE2I, UMR CNRS 6306 , Université de Bourgogne, 21078 Dijon cedex, France*

²*LIRIS, UMR CNRS 5205 , Université Claude Bernard Lyon 1, Université de Lyon, F-69622 , France*

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Abstract

This work establishes the complexity class of several instances of the S -packing coloring problem: for a graph G , a positive integer k and a non decreasing list of integers $S = (s_1, \dots, s_k)$, G is S -colorable, if its vertices can be partitioned into sets S_i , $i = 1, \dots, k$, where each S_i being a s_i -packing (a set of vertices at pairwise distance greater than s_i). For a list of three integers, a dichotomy between NP-complete problems and polynomial time solvable problems is determined for subcubic graphs. Moreover, for an unfixed size of list, the complexity of the S -packing coloring problem is determined for several instances of the problem. These properties are used in order to prove a dichotomy between NP-complete problems and polynomial time solvable problems for lists of at most four integers.

Keywords: NP-complete problem; Distance; Packing chromatic number; S -packing chromatic number; d -distance coloring.

1 Introduction

We consider only undirected connected graphs in this paper. For a graph G , an i -packing is a set $X_i \subseteq V(G)$ such that for any distinct pair $u, v \in X_i$, $d_G(u, v) > i$, where $d_G(u, v)$ denotes the usual shortest path distance between u and v . We will use X_i to refer to an i -packing in a graph G . For a non decreasing sequence of positive integers $S = (s_i, i \in \mathbb{N}^*)$, a S - k -(packing)-coloring of G is a partition of $V(G)$ into sets S_1, \dots, S_k , where each S_i being a s_i -packing. For

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a non decreasing list of k integers $S' = (s_1, \dots, s_k)$, a S' -(*packing*) coloring of G is a S - k -coloring of G for a sequence S which begins with a list S' . A graph G is S -colorable if there exists a S -coloring of G . By $|S|$, we denote the size of a list S , and by s_i we denote the i th element of a list. Let $S = (s_1, s_2, \dots)$ and $S' = (s'_1, s'_2, \dots)$ be two non decreasing lists of integers with $|S| = |S'|$. We have $S \leq S'$ if $s_i \geq s'_i$, for every integer i , $0 \leq i \leq |S|$. Note that if $S \leq S'$, G is S -colorable implies G is S' -colorable. Then we define the following decision problem:

S -COL

Instance : A graph G .

Question: Does G have a S -coloring?

The problem $(1, \dots, 1)$ -COL corresponds to the k -coloring problem (where k is the number of 1) which is known to be NP-complete for $k \geq 3$. The S -coloring problem generalizes coloring problems with distance constraints like the *packing chromatic number* or the *distance chromatic number* of a graph. We denote by P-COL, the problem $(1, 2, \dots, k)$ -COL for an integer k . The packing chromatic number [12] of G is the least integer k such that G is $(1, 2, \dots, k)$ -colorable. A serie of works [4, 7, 9, 12] considered the packing chromatic number of infinite grids. The d -distance chromatic number [18] of G is the least integer k such that G is (d, \dots, d) -colorable, with k the number of integer d in (d, \dots, d) . Initially, the concept of S -coloring has been introduced by Goddard et al. [12] and Fiala et al. [8]. The S -coloring problem was considered in other papers [11, 13, 14]. The S -coloring problem, with $|S| = 3$ has been introduced by Goddard et al. [12] in order to determine the complexity of the packing chromatic number when $k = 4$. Moreover, Goddard and Xu [13] have proven that for $|S| = 3$, S -COL is NP-complete if $s_1 = s_2 = 1$ or if $s_1 = 1$ and $s_2 = s_3 = 2$ and polynomial time solvable otherwise. About the complexity of S -COL, Fiala et al. [8] have proven that P-COL is NP-complete for trees and Argiroffo et al. [2, 1] have proven that P-COL is polynomial time solvable on some classes of graphs. In this article, for a (s_1, s_2, \dots) -coloring of a graph, we prefer to map vertices to the color multi-set $\{s_1, s_2, \dots\}$ even if two colors can then be denoted by the same number. This notation allows the reader to directly see to which type of packing the vertex belong depending on its color. When needed, we will denote colors of vertices in different i -packings by i_a, i_b, \dots . We start this article with the following observation.

Observation 1.1. *For any non decreasing list of integers S , we have S -COL \in NP: for a non decreasing list of integers and a S -colorable graph G as instances, if G is S -colorable, there exists a certificate (a S -coloring) which permit to determine in polynomial time that the graph G is S -colorable.*

In the second section, for a list S of three integers, we determine the family of S -colorable trees. Moreover, we determine dichotomies on cubic graphs, subcubic graphs and bipartite graphs. In the third section, we determine polynomial time solvable and NP-complete instances of S -COL, for unfixed size of lists. In

the last section, a dichotomy between NP-complete instances and polynomial time solvable instances of S -COL is determined for $|S| \leq 4$.

2 Complexity of S -COL for a list of three integers and several classes of graphs

2.1 S -colorable trees, for $|S| = 3$

In [13], the family of S -colorable graphs, for $|S| = 3$, is described in the case S -COL is solved in polynomial time. By adding a leaf on a vertex v , we mean adding a new vertex and connecting it to v . By subdividing an edge uv , we mean removing the edge uv , adding a new vertex and connect it to u and v . The subdivision of a graph G , denoted by $S(G)$, is the graph G in which every edge is subdivided once.

Claim 2.1. *Let S be a non decreasing list of three integers. If $s_2 = 1$, then every tree is S -colorable. Moreover,*

1. *If $s_1 = 3$, then the only S -colorable graphs are paths of length at most 2;*
2. *If $s_1 = 2$, then the only S -colorable graphs are paths of length at most 3;*
3. *If $s_1 = s_2 = 2$ and $s_3 > 2$, then the only S -colorable graphs are paths of length at most 4;*
4. *If $s_1 = s_3 = 2$, then the only S -colorable graphs are paths and cycles of length k , with k a multiple of 3.*

Claim 2.2. *Let S be a non decreasing list of integers, with $|S| = 3$.*

1. *If $s_1 = 1$, $s_2 = 3$ and $s_3 > 3$, the only S -colorable trees are stars where every edge is subdivided once and leaves are added on vertices not obtained by subdivision. If the star is composed of a single edge, the edge can be not subdivided;*
2. *If $s_1 = 1$, $s_2 = 2$ and $s_3 > 3$, the only S -colorable trees are stars where every edge except at most one are subdivided once and leaves are added on vertices not obtained by subdivision.*

By adding a multi-leaf on a vertex x , we mean adding leaves on x and adding leaves on one leaf added previously. By β , we denote the vertex cover number¹.

Proposition 2.3 ([13]). *Let S be a non decreasing list of integers, with $|S| = 3$.*

1. *For $S = (1, 3, 3)$, a tree T is S -colorable, if $\beta(T) \leq 2$ or T is the subdivision of a tree in which leaves are added on the original vertices of the tree;*

¹The vertex cover number of a graph is the minimum size of a set of vertices X such that each edge of the graph is incident to at least one vertex of X .

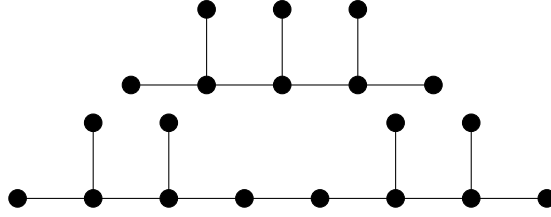


Figure 1: The graphs T_1 (on the top) and T_2 (on the bottom).

2. For $S = (1, 2, 3)$, a tree T with bipartition $T = (A, B)$ is S -colorable, if $\beta(T) \leq 2$ or T is the subdivision of a tree in which leaves are added on the original vertices and a multi-leaf is added on some original vertices that come from A .

Observation 2.4. The trees T_1 and T_2 from Figure 1 are not $(1, 2, 2)$ -colorable.

Proposition 2.5. A tree T is $(1, 2, 2)$ -colorable if and only if it does not contain the tree T_1 or T_2 from Figure 1.

Proof. Using Observation 2.4, if T contains the graph T_1 or T_2 , then T is not $(1, 2, 2)$ -colorable. If T does not contain T_1 , then a vertex of degree at least 3 cannot be adjacent to two vertices of degree at least 3. By a 3-vertex we denote a vertex of degree at least 3. Two 3-vertices are path-connected if there are only vertices of degree 2 in the path between those two vertices. Let a 3a-vertex (a 3b-vertex, respectively) be a 3-vertex adjacent (not adjacent, respectively) to another 3-vertex. We construct a $(1, 2, 2)$ -coloring of T as follows: take a 3-vertex u , give an arbitrary color to this vertex and its possible 3-vertex neighbor and extend the coloring to path-connected 3-vertices. Do the same process from these new colored vertices. Let v_1 and v_2 be path-connected 3-vertices at distance i . Suppose, without loss of generality, that v_1 is colored by 2_a . If i is even then the coloring of v_1 can be extended to v_2 using the pattern $1, 2_b, 1, 2_a, \dots, 1$ for the vertices in the path between v_1 and v_2 . The color 2_a (or 2_b , depending on i) is given to v_2 . If $i > 3$ is odd then the coloring of v_1 can be extended to v_2 using the pattern $1, 2_b, 2_a, 1, 2_b, 1, 2_a, \dots, 1$ for the vertices in the path between v_1 and v_2 . The color 2_a (or 2_b , depending on i) is given to v_2 . If $i = 3$, as T does not contain T_2 , then v_1 and v_2 are not both 3a-vertices. Without loss of generality, suppose v_1 is a 3a-vertex. We give the color 1 and 2_b to the two vertices of the path and the color 2_a to v_2 . \square

3 NP-complete instances of S -COL, for $|S| = 3$

The following result establishes a dichotomy between NP-complete problems and polynomial time solvable problems for $|S| = 3$.

Theorem 3.1 ([13]). *Let $k \geq 1$ be an integer. The problems $(1, 1, k)$ -COL and $(1, 2, 2)$ -COL are both NP-complete. Except these problems, S -COL is polynomial time solvable for $|S| = 3$.*

This result was found by Goddard and Xu [13] and independently by us. For $(1, 1, k)$ -COL, the authors of [13] did a reduction from 3-COL with a graph of maximal degree Δ , the produced graph has maximal degree 2Δ . As 3-COL is polynomial time solvable for small Δ , the proof cannot be easily changed to have a reduction with subcubic graphs. For $(1, 2, 2)$ -COL, the authors did a reduction from NAE SAT, the produced graph has maximal degree Δ , Δ being the maximal number of times a variable can appear positively or negatively. This reduction cannot be easily changed to have a reduction with subcubic graphs. As our transformation from [10] gives us subcubic graphs, we establish the same result for subcubic graphs.

Definition 3.1. *Let S be a non decreasing list of three integers. Let s_i and s_j be two integers in this list.*

A s_i - s_j -transmitter (respectively, a s_i - s_j -antitransmitter) is a graph L such that there exist two vertices u and v called connectors and in every S -coloring of L , u and v have a same color among $\{s_i, s_j\}$ (respectively, different colors among $\{s_i, s_j\}$). Moreover, there should exist a S -coloring of L such that every vertex that have a different color c that of u or v must be at distance greater than $\lfloor c/2 \rfloor$ from u or v .

A s_i - s_j -clause simulator is a graph H such that there exist three vertices u_1 , u_2 and u_3 called connectors, and in every S -coloring of H , u_1 , u_2 and u_3 have colors among $\{s_i, s_j\}$ and never u_1 , u_2 and u_3 have the same color. Moreover, there should exist S -colorings of H such that every vertex that have a different color c that of u_1 or u_2 or u_3 must be at distance greater than $\lfloor c/2 \rfloor$ from u_1 , u_2 or u_3 , for any possible configuration of colors among $\{s_i, s_j\}$ for the connectors.

Lemma 3.2. *Let S be a non decreasing list of three integers and let s_i and s_j be two integers in this list. If there exists a s_i - s_j -transmitter L , a s_i - s_j -antitransmitter J and a s_i - s_j -clause simulator H , then S -COL is NP-complete.*

Proof. Suppose there exists a transmitter L , an antitransmitter J and a clause simulator H . Connecting two vertices u and v by L (respectively, by J) corresponds to add vertices and edges in such a way that u and v become the connectors of L (respectively, of J). The original vertices correspond to the vertices u and v . Let a L - k -cycle be a cycle of order k in which each edge is replaced by L (it means that if uv is an edge, we remove the edge and connect u and v by L). The proof is by reduction from NAE 3SAT:

a literal is either a variable x (a positive occurrence of x) or the negation of a variable (a negative occurrence of x). A clause is a disjunction of literals and the size of a clause is its number of literals. A truth assignment gives values (true or false) for each variable. For a given truth assignment on a set of clauses, if there is a true literal and a false literal in each clause, then it satisfies not equally the set of clauses.

Not-all-equal 3-satisfiability (NAE 3SAT)

Instance: Collection $C = [c_1, c_2, \dots, c_m]$ of clauses on a finite set of variables of size n such that $|c_i| = 3$ for $1 \leq i \leq m$.

Question: Is there a truth assignment for variables that satisfies not equally all clauses in C ?

Let C be a collection of clauses. Let m be the number of clauses and let n be the number of variables. We construct a graph G as follows. For each clause, we associate a clause simulator H with connectors ℓ_1 , ℓ_2 and ℓ_3 representing the three literals of the clause. For each variable x with p_1 positive occurrences of x and p_2 negative occurrences of x , we associate one $L - (p_1 + 1)$ -cycle and one $L - (p_2 + 1)$ -cycle where one connector b_1 of the $L - (p_1 + 1)$ -cycle and one connector b_2 of the $L - (p_2 + 1)$ -cycle are connected by J . For each literal ℓ'_1 which is a positive occurrence of x (respectively, negative occurrence of x), we connect ℓ'_1 by L to a connector of the $L - (p_1 + 1)$ -cycle different from b_1 (respectively, to a connector of the $L - (p_2 + 1)$ -cycle different from b_2), so that every connector of the $L - (p_1 + 1)$ -cycle (respectively, $L - (p_2 + 1)$) is connected to only one occurrence of x .

Suppose there exists a truth assignment of variables that satisfies not equally all clauses in C . To each clause c corresponds a clause simulator H . Given a truth assignment, we give the color s_i to the connectors representing a true literal and s_j to the connectors representing a false literal. If a connector of a transmitter or an antitransmitter is colored, then we give the other possible color to the other connector. Using the existing S -coloring of H , L and J , it gives us a S -coloring of G (the vertices colored have enough distance between them by hypothesis).

Reciprocally, suppose there exists a S -coloring of G . For each H , the connectors of H have colors among $\{s_i, s_j\}$, by hypothesis. By construction, positive occurrences and negative occurrences of a same variable must have different colors. For each positive or negative occurrence of a variable x , we create a truth assignment by giving the value true (respectively, false) to positive occurrences colored by s_i (respectively, s_j) and by giving the value false (respectively, true) to negative occurrences colored by s_i (respectively, s_j). By construction, this truth assignment satisfies not equally all clauses in C . \square

Note that if H , L and J are subcubic graphs, the connectors of H have degree 2 and if the connectors of L and J have degree 1, then the constructed graph is subcubic. Moreover, if H , L and J are bipartite and any pair of connectors of L , J and H are connected by a path of even length, then the constructed graph is bipartite.

Theorem 3.3. *The problem $(1, 2, 2)$ -COL is NP-complete for subcubic bipartite graphs.*

Proof. Let L and J be the subcubic graphs from Figure 2 with the corresponding connectors ℓ'_1 and ℓ'_2 . Remark that if a vertex has degree 3, then it cannot be colored 1, as the coloring cannot be extended to a $(1, 2, 2)$ -coloring. Therefore,

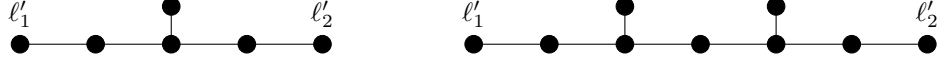


Figure 2: The graph L (on the left) and the graph J (on the right).

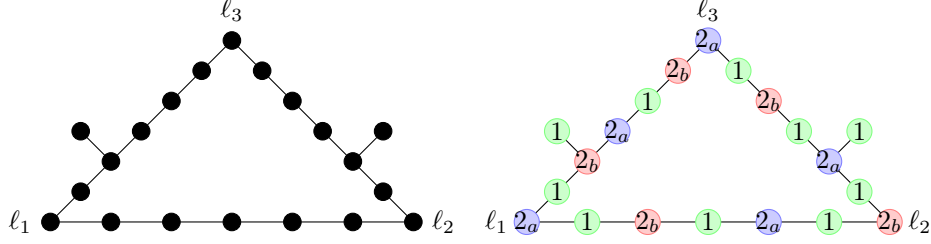


Figure 3: The graph H (on the left) and a $(1, 2, 2)$ -coloring of H (on the right).

every connector must be colored by 2_a or 2_b , as in the final constructed graph they will have degree 3. Observe that there are vertices of degree 3 in L and J . Hence, these vertices must also be colored by 2_a or 2_b . If the two connectors of L have different colors, then the central vertex of degree 3 cannot be colored. Thus, in every $(1, 2, 2)$ -coloring of L , the connectors must have the same color. A $(1, 2, 2)$ -coloring of L exists giving the color 2_a to the connectors, 2_b to the central vertex of degree 3 and 1 to the remaining vertices.

If the two connectors of J have the same color, then the two central vertices of degree 3 cannot be colored. Thus, in every $(1, 2, 2)$ -coloring of J , the connectors must be colored differently. A $(1, 2, 2)$ -coloring of J exists giving the color 2_a to a connector and to a central vertex of degree 3, 2_b to the other connector and to the other central vertex of degree 3 and 1 to the remaining vertices. To conclude, L is a 2-2-transmitter and J is a 2-2-antitransmitter, respectively.

Let H be the subcubic graph from the left part of Figure 3 with the corresponding connectors ℓ_1 , ℓ_2 and ℓ_3 . The graph H contains two vertices of degree 3 and three connectors. Moreover, these vertices should be colored by 2_a or 2_b . Suppose ℓ_1 , ℓ_2 and ℓ_3 have the same color 2_a . The two vertices of degree 3 should be colored by 2_b . Observe that the vertices at distance 2 from ℓ_3 could only have the color 1. Thus, the two neighbors of ℓ_3 cannot be colored as they cannot be colored 1 or 2_a and they cannot have both the color 2_b . Therefore, in every $(1, 2, 2)$ -coloring of H the connectors must be colored differently. The right part of Figure 3 gives the existence of a S -coloring with the suited property for ℓ_1 and ℓ_3 with the same color (for the other cases, the $(1, 2, 2)$ -coloring is similar). To conclude, H is a 2-2-clause simulator. \square

Note that no r -regular graph, for $r \geq 3$, is $(1, 2, 2)$ -colorable. Thus, this problem seems to be hard only on graphs containing a large proportion of vertices of degree at most 2.

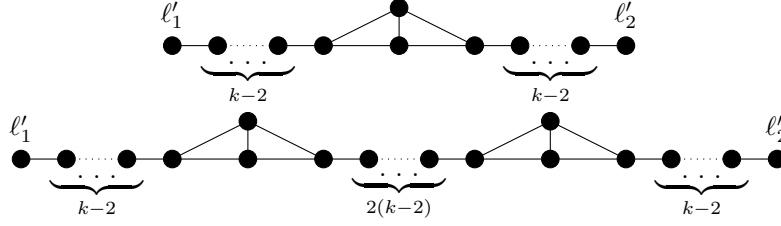


Figure 4: The graph L_k (on the top) and the graph J_k (on the bottom).

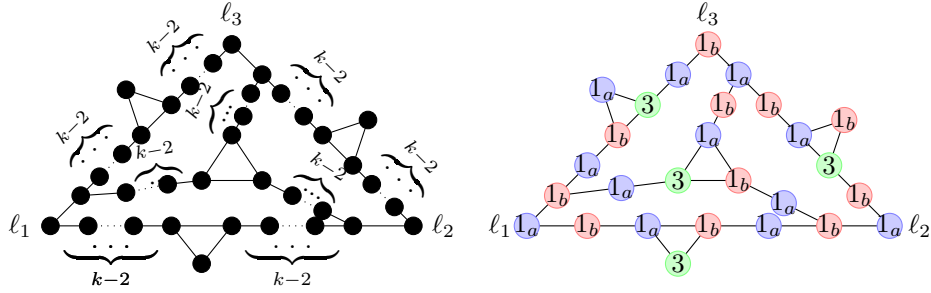


Figure 5: The graph H_k (on the left) and a $(1, 1, 3)$ -coloring of H_3 (on the right).

Theorem 3.4. *Let $k \geq 2$ be a positive integer. The problem $(1, 1, k)$ -COL is NP-complete for subcubic graphs.*

Proof. Let L_k and J_k be the subcubic graphs from Figure 4 with the corresponding connectors ℓ'_1 and ℓ'_2 . First, observe that there are two triangles in L_k . If no common vertices of the two triangles are colored by k , then the coloring cannot be extended to a $(1, 1, k)$ -coloring of L_k . Thus, in every $(1, 1, k)$ -coloring one of these two vertices is colored by k and the remaining vertices could only be colored by 1_a or 1_b , alternating the colors. Therefore, because the number of the remaining vertices is odd, if one of the connector is colored one color 1, the other one could only be colored by the same color 1. In this $(1, 1, k)$ -coloring, the vertices colored by k are at distance k from the connectors.

Observe that there are two pairs of adjacent triangles in J . Thus, for the same reason than for L , for each pair of adjacent triangles, one of the two vertices which is common to both triangles should be colored by k . Therefore, because the number of remaining vertices is even, if one of the connector is colored one color 1, the other one could only be colored the other color 1. To conclude, L_k is a 1-1-transmitter and J_k is a 1-1-antitransmitter.

Let H_k be the subcubic graph from the left part of Figure 5 with the corresponding connectors ℓ_1 , ℓ_2 and ℓ_3 . The graph H_k contains four triangles and in

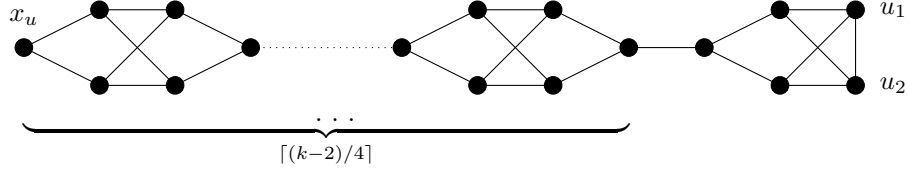


Figure 6: The graph N_k .

every triangle a vertex should be colored by k . Thus, in every $(1, 1, k)$ -coloring the remaining vertices could only be colored by 1_a or 1_b alternating the colors. Without loss of generality, suppose ℓ_1, ℓ_2 and ℓ_3 are colored by the same color 1_a . The two neighbors of these vertices should be colored by 1_b and alternating the colors, the central triangle has its neighbors all colored by the same color 1_b or 1_a (depending on k). Hence, the central triangle cannot be colored, the color of the neighbors being not available. The right part of Figure 5 gives the existence of a S -coloring with the suited property for $k = 3$ (for any other integer k , the coloring is similar). To conclude, H_k is a 1-1-clause simulator. \square

Corollary 3.5. *Let $k \geq 2$ be a positive integer. The problem $(1, 1, k)$ -COL is NP-complete for cubic graphs.*

Proof. Let G be a graph, and G' be the graph obtained with the reduction from Lemma 3.2 using the 1-1-transmitter, 1-1-anititransmitter and 1-1-clause simulator from Proposition 3.4. For each vertex $u \in V(G')$ of degree 2, we add a graph N_k from Figure 6 and connect u to the vertex x_u of N_k . We obtain a cubic graph in the previous reduction. This gadget is clearly $(1, 1, k)$ -colorable, the color k is used to color one vertex among u_1 and u_2 . \square

For different instances of S -COL, with $|S| = 3$, Table 1 summarizes the class of complexity of S -COL for different classes of graphs. We recall that every bipartite graph is $(1, 1, k)$ -colorable, for $k \geq 1$ and that every subcubic graph except K_4 is $(1, 1, 1)$ -colorable by Brooks' theorem.

Class of graphs	$(1, 1, 1)$ -COL	$(1, 2, 2)$ -COL	$(1, 1, k)$ -COL, $(k \geq 2)$
Arbitrary graphs	NP-complete	NP-complete	NP-complete
Subcubic graphs	Polynomial	NP-complete	NP-complete
Cubic graphs	Polynomial	Polynomial	NP-complete
Bipartite graphs	Polynomial	NP-complete	Polynomial
Trees	Polynomial	Polynomial	Polynomial

Table 1: Complexity class of several instances of S -COL.

4 Complexity class of few instances of S -COL, $|S| > 3$

4.1 polynomial time solvable instances of S -COL

We recall the following theorem from Sharp [18] about the complexity of S -COL for a list $S_{d^k} = (d, \dots, d)$, with $|S_{d^k}| = k$.

Theorem 4.1 ([18]). *The problem S_{d^k} -COL is NP-complete for $k \geq \lceil 3d/2 \rceil$ and polynomial time solvable in the other cases.*

The following theorem is a result from Bienstock *et al.* [3]. The definitions of pathwidth, treewidth and minors are not given here, the reader can find them in [16, 19]. Note that for a graph G , the treewidth of G is bounded by the pathwidth of G .

Theorem 4.2 ([3]). *For any forest F , every graph without F as minor has a pathwidth bounded by $|V(F)| - 2$.*

We will use the following theorem from Fiala *et al.* [8] in order to prove that several instances of S -COL are polynomial time solvable.

Theorem 4.3 ([8]). *Let S be a non decreasing list of integers. The problem S -COL is polynomial time solvable for graphs of bounded treewidth.*

This result has been generalized to the decision problem associated with the S - K -coloring problem, for a sequence S which contains only integers smaller than a constant [8]. The theorem from Fiala *et al.* [8] use the result of Courcelle [5] about properties definable in monadic second-order logic.

Proposition 4.4. *Let S be a non decreasing list with $s_1 > 1$ and let G be a graph. If G contains a vertex of degree at least $|S|$, then G is not S -colorable.*

Proof. Let v be a vertex of degree at least $|S|$. As the set $N(v) \cup v$ has diameter at most two, every vertex should be colored differently. However, we have $|N(v) \cup v| \geq |S| + 1$. \square

Definition 4.1. *Let k and d be two positive integers. The Moore bound [15] $n(k, d)$ is the maximal order for a graph of diameter d and maximal degree k .*

Proposition 4.5. *Let S be a non decreasing list with $s_1 > 1$. If there exists an infinite S -colorable graph, then P_∞ is S -colorable.*

Proof. If P_∞ is not S -colorable, then there exists an integer k such that every graph with diameter at least k is not S -colorable. Using Proposition 4.4, note that a S -colorable graph has maximal degree bounded by $|S| - 1$. Thus, using the Moore Bound, every graph of bounded degree and bounded diameter has order bounded by $n(|S| - 1, d)$. \square

Definition 4.2. Let S be a non decreasing list of integers. We define $t_{\max}(S)$ as the maximal order of a S -colorable graph. If $t_{\max}(S)$ is not bounded, then we set $t_{\max}(S) = \infty$.

We have the following observations about t_{\max} :

Observation 4.6. Let S be a non decreasing list of integers.

- If $t_{\max}(S)$ is bounded, then there exists a S -colorable tree of order $t_{\max}(S)$.
- If $s_1 \geq |S|$, then $t_{\max}(S) = |S|$.
- We have $p_{\max}(S) \leq t_{\max}(S)$, with p_{\max} the maximal order of a S -colorable path. Moreover, if $p_{\max}(S) < \infty$ and $s_1 > 2$, then we have $t_{\max}(S) \leq n(|S| - 1, p_{\max}(S))$, using the Moore bound.

Definition 4.3. Let Y_n be the graph with vertex set $V(Y_n) = \{a_i | i \in \{0, \dots, n\} \cup \{b_i | i \in \{1, \dots, n\}\} \cup \{c_i | i \in \{1, \dots, n\}\}$ and edge set $E(Y_n) = \{a_0b_1, a_0c_1\} \cup \{a_ia_j | |i-j|=1\} \cup \{b_ib_j | |i-j|=1\} \cup \{c_ic_j | |i-j|=1\}$. Let Y_∞ be the infinite version of Y_n .

We start with the following observation:

Observation 4.7. Let G be a graph. If G contains Y_n as minor, then G contains the graph Y_n as subgraph.

Proposition 4.8. If Y_∞ is not S -colorable, then S -COL is polynomial time solvable.

Proof. If Y_∞ is not S -colorable, there exists an integer n such that Y_n is not S -colorable (using the De Bruijn-Erdős theorem [6]). Moreover, every graph containing the graph Y_n as minor is not S -colorable, by Observation 4.7. Using Theorem 4.2, every S -colorable graph has treewidth bounded by $3n - 1$. Using Theorem 4.3, S -COL is polynomial time solvable. \square

Note that if $t_{\max}(S) < \infty$, for a non decreasing list of integers S , then there exists an integer n such that every path of diameter n is not S -colorable. Thus, Y_∞ is not S -colorable. We obtain the following Corollary:

Corollary 4.9. If $t_{\max}(S) < \infty$, then S -COL is polynomial time solvable.

Proposition 4.10. Let S be a non decreasing list of integers and let $i > 1$ be an integer. If $s_1 = i$, then S -COL is polynomial time solvable for $|S| < \lceil 3i/2 \rceil$.

Proof. Using Theorem 4.1, the problem S_{d^k} -COL (S_{d^k} is the list composed of k integers d) is polynomial time solvable for $k < \lceil 3i/2 \rceil$. Every $S_{s_1^{|S|}}$ -colorable graphs has bounded treewidth [17] and every S -colorable graph is also $S_{s_1^{|S|}}$ -colorable. Hence, S -colorable graphs have bounded treewidth. Using Theorem 4.3, S -COL is polynomial time solvable. \square

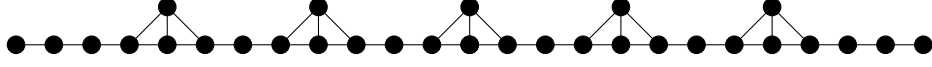


Figure 7: The graph $L_{2,(2,2,2,3)}$.

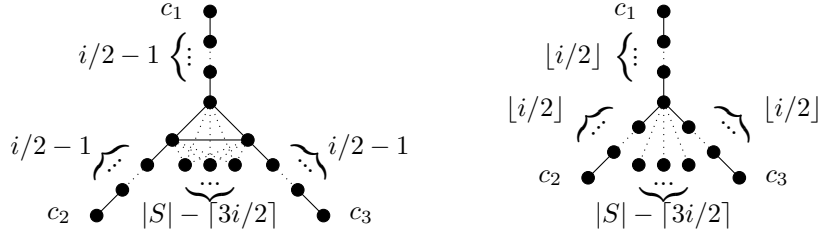


Figure 8: The graphs $R_{i,S}$, for an even i (on the left) and for an odd i (on the right).

4.2 NP-complete instances of S -COL for $|S| \geq 3$

The following proposition is a generalization of the Theorem from Sharp [18, 17].

Theorem 4.11. *The problem S -COL is NP-complete for every list S such that $s_1 = s_{i+1} = i$, with $i > 1$ and $|S| \geq \lceil 3i/2 \rceil$.*

Proof. Let $N_i(S)$ be the number of occurrence of an integer i in the list S and let $n_i^-(S) = N_i(S) - i - 1$. The proof use a reduction from $N_i(S)$ -COL (as $N_i(S) \geq 3$, $N_i(S)$ -COL is NP-complete).

k -COL

Instance: A graph G and an integer k .

Question: Is the graph G k -colorable?

Let G be a graph. We set $m = \prod_{j \in \mathbb{N}^* \mid |N_j(S)| \geq 1} (j + 1)$. We construct a graph

$L_{i,S}$ composed of vertices $u_{j,l}$, for integers j and l , with $-m \leq j \leq m$. Let j, j', l and l' be four integers. Two vertices $u_{j,l}$ and $u_{j',l'}$ in $V(L_{i,S})$ are adjacent if and only if $|j - j'| \leq 1$ (except if $u_{j,l} = u_{j',l'}$). The rank of a vertex is $\text{rank}(u_{j,l}) = j$ and corresponds to its position in $L_{i,S}$. Let j be an integer and let $\text{rank}_{\max}(j) = \max\{l \mid u_{j,l} \in V(L_{i,S})\}$. Let k be an integer, a k -insertion at rank j consists in adding vertices $u_{j, \text{rank}_{\max}(j)+1}, \dots, u_{j, \text{rank}_{\max}(j)+k}$ in the graph $L_{i,S}$.

We construct the graph $L_{i,S}$ as follows: the graph $L_{i,S}$ always contains vertices $u_{-m,0}, \dots, u_{m,0}$. Depending the list S , we have to do the following insertions:

1. We do a $n_i^-(S)$ -insertion at rank j , for each integer j such that $j \equiv \lfloor \frac{i+1}{2} \rfloor \pmod{i+1}$ and $-m \leq j \leq m$.

2. For every integer $k > i$, we do a $N_k(S)$ -insertion at rank j , for each integer j such that $j \equiv 0 \pmod{k+1}$ and $-m < j < m$.

The last operation consists to add edges in the graph in order that the adjacency property (two vertices of rank j and j' are adjacent if and only if $|j - j'| \leq 1$) is verified. The connectors of a graph are specified vertices of degree 1. The connectors of $L_{i,S}$ are the vertices $u_{-m,0}$ and $u_{m,0}$. The graph $L_{2,(2,2,2,3)}$ is illustrated in Figure 7.

Let $R_{i,S}$ be the graph from Figure 8. Note that depending the parity of i , the graph $R_{i,S}$ is different. The connectors of $R_{i,S}$ are the vertices c_1 , c_2 and c_3 from Figure 8. Connecting a graph G to another graph G' is an operation that consists in merging one connector of degree 1 of G and a connector of degree 1 of G' . In order to construct the graph $J_{i,S}$, we begin with constructing a graph from two copies of $L_{i,S}$ and a copy of $R_{i,S}$ where we connect $R_{i,S}$ to the two copies of $L_{i,S}$. Suppose that u and v are the two connectors of degree 1 of the copies of $L_{i,S}$ in the constructed graph. We finish with deleting one vertex among u and v . If $i = 2$, the following property should be verified in the copy of $L_{i,S}$: $\text{rang}_{\max}(w) = 1$, for w the vertex adjacent to the removed vertex. The constructed graph is the graph $J_{i,S}$. The connectors of this graph are the vertices w and the remaining vertex among u and v .

Let d be a positive integer. The graph G_d is the graph obtained from $R_{i,S}$ and $L_{i,S}$, as follows: we begin with taking d copies L_0, \dots, L_{d-1} of $L_{i,S}$ and d copies R_0, \dots, R_{d-1} of $R_{i,S}$. For every integer k , with $0 \leq k \leq d-1$, we connect R_k to L_k and $L_{k+1 \pmod{d}}$. The obtained graph is the graph G_d and have d connectors, these connectors are the remaining connectors of degree 1 of each copy of $R_{i,S}$.

Finally, we construct the graph G' from the original graph G , as follows: for every vertex $v \in V(G)$, we associate a graph G_v which is a copy of $G_{d(v)}$. We call original vertices, the connectors of a graph G_v , for a vertex v (even if these vertices do not have degree degree 1 in the constructed graph). For every edge $uv \in E(G)$, we connect G_v and G_u by a graph $J_{i,S}$.

In order to prove that the following construction is a reduction from $N_i(S)$ -COL to S -COL, we prove the following facts:

Fact 1: The graph $R_{i,S}$ is S -colorable and in every S -coloring of $R_{i,S}$, the connectors of $R_{i,S}$ have the same color i .

Proof: The diameter of $R_{i,S}$ is $i+1$ and there are only three vertices which are at pairwise distance at least $i+1$, these vertices are the three connectors of $R_{i,S}$. As $|V(R_{i,S})| = |S|+2$, the connectors must have the same color. Moreover, this color must be a color i . There exists a S -coloring of $R_{i,S}$ consisting in giving the same color i to the connectors and giving a different color to each remaining vertex.

Fact 2: For every S -coloring of an induced $L_{i,S}$ in G' and for every color $s_k \neq i$, there is a vertex of rank j colored by s_k , for every integer j such that $j \equiv 0 \pmod{s_k+1}$ and $-m < j < m$.

Proof: There are $\sum_{j=i+1}^{|S|} N_j(S) + 1 = |S| - n_i^-(S)$ vertices of rank 0. Suppose

that at least two vertices of rank 0 are colored by some colors i . We have to use each color i two times in order to color the vertices of rank ℓ , for $-(i+1) < \ell \leq (i+1)$, as there are $|S| + N_i(S)$ vertices of such rank. It is not possible to use each color i two times if two vertices of rank 0 are colored by some color i . Thus, there are vertices of rank 0 of each color $s_k \neq i$.

By induction on j , for $-m \leq j \leq -(i+1)$ and $(i+1) < j \leq m$, we can easily prove that for every color $s_k \neq i$, there is a vertex of rank j colored by s_k , for every integer j such that $j \equiv 0 \pmod{s_k+1}$.

Fact 3: In every S -coloring of $L_{i,S}$ and $J_{i,S}$, the connectors have the same color i in $L_{i,S}$ and two different colors i in $J_{i,S}$.

Proof: Using Fact 2, note that in every S -coloring of $L_{i,S}$ and $J_{i,S}$ there is a vertex of rank j colored by s_k , $s_k \neq i$, for every integer j such that $j \equiv 0 \pmod{s_k+1}$ and $-m < j < m$. Consequently, there is a vertex colored by s_k at distance at most s_k from the connectors of $L_{i,S}$ and $J_{i,S}$. Note also that $-m$ and m have both 0 as remainder modulo $i+1$ in $L_{i,S}$. There is only one vertex colored i of rank j , for each j such that $j \equiv 0 \pmod{m}$, this color i is the same for each such vertex. Hence, the two connectors of $L_{i,S}$ have the same color i . The argument is similar for $J_{i,S}$ (we have to use Fact 1 also).

Fact 4: There exists a S -coloring of $L_{i,S}$ and $J_{i,S}$.

Proof: First, we color the vertices $u_{0,k}$, for $-m \leq k \leq m$, alternating $i+1$ colors i . Second, for every color $s_k \neq i$, we color the remaining vertices of rank j with color s_k , for every integer j such that $j \equiv 0 \pmod{s_k+1}$ and $-m < j < m$. Finally, we color the remaining vertices of rank j with n_i^- remaining colors i , for $j \equiv \lceil i/2 \rceil \pmod{i+1}$ and $-m \leq j \leq m$. The coloring is similar for $J_{i,S}$.

Fact 5: Let d be a positive integer. In every S -coloring of G_d , the original vertices have the same color i .

Proof: This fact is a consequence of Facts 1 and 3.

Fact 6: For every pair of colors (i_a, i_b) in $N_i(S)$, with $i_a \neq i_b$, there exists a S -coloring of an induced $J_{i,S}$ in G' , with connectors of color i_a and i_b .

Proof: Let L^1 be the copy of $L_{i,S}$ in which no vertex has been removed and let L^2 be the second copy of $L_{i,S}$. We begin with using a list of $i+1$ different colors i to color the vertices $u_{0,k}$, for $-m \leq k \leq m$, in L^1 . We suppose that the color i_a is used to color the initial connectors of the copy of $R_{i,S}$. We give the color i_b to the neighbor of the connector of the copy of $R_{i,S}$ which is a connector of L^2 . We claim that we can extend the coloring to the vertices of the copy of $R_{i,S}$. Using colorings from Fact 4, we can also extend the coloring to the whole graph (with colors i_a and i_b for the connectors).

Suppose that the graph G is $N_i(S)$ -colorable, with colors $1, \dots, N_i(S)$. For every vertex v of color j , for $1 \leq j \leq N_i(S)$, we use a color s_j in order to color the original vertices of G_v in the graph G' . Using Facts 1, 4 et 6, we can extend the coloring to a S -coloring of the whole graph.

Suppose that G' is S -colorable. Using Fact 5, for each vertex $v \in V(G)$, the original vertices of G_v have the same color i . We conclude that we can construct a $N_i(S)$ -coloring of G using the color j to color vertices with color s_j in G' , for $1 \leq j \leq N_i(S)$. \square

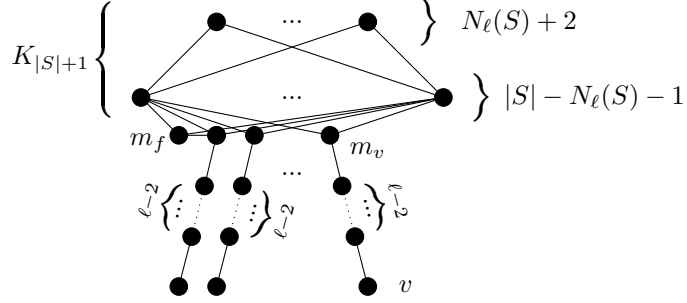


Figure 9: Illustration of the reduction of Proposition 4.13.

Corollary 4.12. *Let k , k' and k'' be positive integers. The problem S -COL is NP-complete for the lists $S = (2, 2, 2, k)$ and $S' = (4, 4, 4, 4, k', k'')$.*

Proposition 4.13. *Let S be a non decreasing list of integers with $s_2 = 1$ and $|S| \geq 3$. If S -COL is NP-complete, then S' -COL is NP-complete for $|S'| = |S| + 1$, $s'_{|S|+1} \geq s_{|S|}$ and $s'_i = s_i$, for each integer i , with $1 \leq i \leq |S|$.*

Proof. Suppose that S -COL is NP-complete and that $s'_{|S|+1} = \ell$. From a graph G , we construct a new graph G' as follows: for every vertex $v \in V(G)$, we begin with connecting v to the extremity of a path of $\ell - 1$ vertices, with m_v the other extremity of the path (in the case $\ell = 2$, the path has only one vertex and the extremities denote the same vertex). Thus, we add a new vertex m_f that we connect to a vertex m_v , for some v . Afterwards, we add a copy of $K_{|S|+1}$ and we do the join between the set $\{m_v | v \in V(G)\} \cup \{m_f\}$ and a set of $|S| - N_\ell(S) - 1$ vertices in $K_{|S|+1}$. The obtained graph is the graph G' . The original vertices of G' are the vertices that come from the graph G .

Suppose that G is S -colorable. We use a S -coloring of G in order to color the original vertices of G' . Thus, for every vertex v , we alternate the colors $s_1 = 1_a$ and $s_2 = 1_b$ in order to color every path until m_v . Afterwards, we give the color 1_a or 1_b to m_f (depending the color of the vertex adjacent to m_f). We have $N_\ell(S) + 2$ vertices in $K_{|S|+1}$ non adjacent to m_f . The colors 1_a , 1_b and the maximum number of color ℓ in S are given to these vertices. The remaining colors are given to the non colored vertices in $K_{|S|+1}$.

Suppose that G' is S' -colorable. Every vertex of $K_{|S|+1}$ should be colored differently. The two colors given to m_v and m_f should be used in order to color the $N_\ell(S) + 2$ vertices in $K_{|S|+1}$ non adjacent to m_f . Thus, m_v and m_f have some colors 1. At most $N_\ell(S)$ vertices can be colored by ℓ among these $N_\ell(S) + 2$ vertices in $K_{|S|+1}$ non adjacent to m_f . Consequently, a vertex should be colored by a color ℓ among the remaining vertices of $K_{|S|+1}$. This color ℓ cannot be used in order to color the other vertices of G' , the other vertices being at distance less than ℓ of a vertex colored by this color ℓ . The coloring of the original vertices of G' is used in order to create a S -coloring of G . \square

Corollary 4.14. *The problem S -COL is NP-complete for every list S such that $s_1 = s_2 = 1$ and $|S| \geq 3$.*

Proposition 4.15. *Let S be a non decreasing list of integers. If S -COL is NP-complete, then S' -COL is NP-complete, for every list S' such that $|S'| = |S| + 1$, $s'_1 = 1$ and $s'_{i+1} = 2s_i$ or $s'_{i+1} = 2s_i + 1$, for every integer i , with $1 \leq i \leq |S|$.*

Proof. Suppose that S -COL is NP-complete. There exists a reduction from S -COL to S' -COL similar with the reduction introduced by Goddard *et al.* [12] from $(1, 1, 2)$ -COL to $(1, 2, 3, 4)$ -COL.

Let G be a connected graph. We construct a new graph G' from G by replacing each edge uv by $|S| + 1$ parallel edges that we subdivide. The vertices u and v are called original vertices.

Suppose that G is S -colorable. Let $X_i \subseteq V(G)$ be an i -packing. Every pair of vertices (u, v) in X_i is such that $d(u, v) > i$. Let $X'_i \subseteq V(G')$ be the set of original vertices in X_i . Observe that every pair of vertices (u, v) in X'_i is such that $d_{G'}(u, v) > 2i + 1$. Thus, X'_i is a $(2i + 1)$ -packing (and consequently a $2i$ -packing). Let X'_1 be the set of non original vertices (created by subdivision). This set is independent and we color the vertices in this set by color 1. We give the color s'_{i+1} to the original vertices of G colored by s_i , for $1 \leq i \leq |S|$, we obtain a S' -coloring of G' .

Suppose that G' is S' -colorable and suppose that there exists an original vertex u colored by 1. As u has $|S| + 1$ neighbors, two of its neighbors have the same color s'_i . Thus, no original vertex is colored by 1. Using the S' -coloring of the original vertices of G' , we can create a S -coloring of G by giving the color s_{i-1} to the vertices colored by s'_i in G' , for $2 \leq i \leq |S| + 1$. \square

Corollary 4.16. *The problem S -COL is NP-complete for every list such that $|S| = 4$ and $S \geq S'$, with $S' = (1, 3, 3, 3)$ or $S' = (1, 3, 5, 5)$.*

5 Dichotomy for $|S| = 4$

In this section, we prove a dichotomy between NP-complete problems and polynomial time solvable problems, using the results from the previous section.

Observation 5.1. *Let S be a non decreasing list of integers and let n be an integer. The graph Y_n is not S -colorable for:*

1. $S = (2, 3, 3, 3)$ and $n = 2$;
2. $S = (2, 2, 3, 4)$ and $n = 4$;
3. $S = (1, 4, 4, 4)$ and $n = 3$.

Observation 5.2. *The graph P_{16} is not $(1, 3, 5, 8)$ -colorable.*

In the following propositions, we subdivide three times an edge uv . For an edge subdivided three times, the central subdivided vertex is the vertex which is not a neighbor of u or v :

1. A *2-add* on a vertex u is an operation which consists in adding a vertex w_u and an independent set of vertices I , adding the edge uw_u and adding some edges between $\{u, w_u\}$ and the vertices of I .
2. A *3-add* on a vertex u is an operation which consists in adding a vertex w_u and an independent set of vertices I and adding some edges between $\{u, w_u\}$ and the vertices of I .
3. A *5-3-add* on a vertex u is an operation which consists in adding two vertices w_u and w'_u and two independent sets of vertices I and I' , adding some edges between $\{u, w_u\}$ and the vertices of I , between $\{w_u, w'_u\}$ and the vertices of I' and eventually adding the edge $w_u w'_u$.
4. A *5-2-add* on a vertex u is an operation which consists in adding three vertices w_u , w'_u and w''_u and three independent sets of vertices I , I' and I'' and adding the edges uw_u and $w_u w'_u$ and adding some edges between $\{u, w_u\}$ and the vertices of I , between $\{u, w'_u\}$ and the vertices of I' and between $\{w'_u, w''_u\}$ and the vertices of I'' .
5. A *6-3-add* on a vertex u is an operation which consists in adding two vertices w_u and w'_u and two independent sets of vertices I and I' , adding the edge $w_u w'_u$ and adding some edges between $\{u, w_u\}$ and the vertices of I and between $\{w_u, w'_u\}$ and the vertices of I' .
6. A *6-2a-add* on a vertex u is an operation which consists in adding three vertices w_u , w'_u and w''_u and three independent sets of vertices I , I' and I'' , adding the edge $w_u w'_u$ and adding some edges between $\{u, w_u\}$ and the vertices of I , between $\{w_u, w'_u\}$ and the vertices of I' and between $\{w'_u, w''_u\}$ and the vertices of I'' .
7. A *6-2b-add* on a vertex u is an operation which consists in adding three vertices w_u , w'_u and w''_u and three independent sets of vertices I , I' and I'' , adding the edge uw_u , adding some edges between $\{u, w_u\}$ and the vertices of I , between $\{w_u, w'_u\}$ and the vertices of I' and between $\{w'_u, w''_u\}$ and the vertices of I'' and eventually adding the edge $w'_u w''_u$.

These operations are illustrated in Figure 10.

Theorem 5.3. *Let $S = (1, 3, k, k')$, with $5 \leq k \leq 7$ and $6 \leq k' \leq 7$. A graph G is S -colorable if and only if one of these three following conditions is true:*

1. $\beta(G) \leq 3$;
2. $\beta(G) = 4$ and there exist two vertices u and v in a vertex cover of size 4 such that $d(u, v) \geq 4$;
3. the graph G is obtained from a bipartite multi-graph $G' = (A, B)$ in which every edge is subdivided three times and where we can do the following operations:

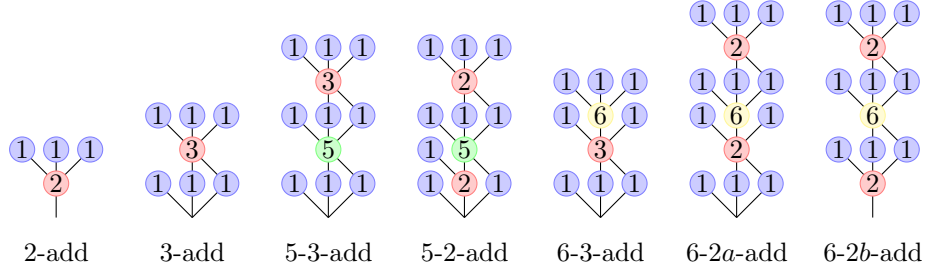


Figure 10: The different used operations in Propositions 5.3 and 5.4.

- (a) we can add leaves on the central subdivided vertices and do a 3-add or adding leaves on vertices that come from A or B ;
- (b) if $k \leq 6$, we can do a 6-3-add on vertices that come from A . If $k' \leq 6$, we can do a 6-3-add on the vertices that come from A and B ;
- (c) if $k = 5$, instead of doing a 6-3-add on vertices that come from A , we can do a 5-3-add or several 6-3-adds.

Proof. Let G be a graph for which one of the three conditions is true. If the first condition is true, then we color the vertices of the vertex cover by colors 3, k and k' and the remaining vertices by color 1. If the second condition is true, then we color u and v by color 3, the other vertices of the vertex cover are colored by colors k and k' and the remaining vertices are colored by 1. If the third condition is true, then we color the central subdivided vertices by color 3, the vertices that come from A by color k' , the vertices that come from B by color k , the vertices that have been added as in Figure 10 and the remaining vertices by color 1. To conclude, if one of the three conditions is true, then G is S -colorable.

Suppose that G is S -colorable. If no vertex is colored by k or k' , then the first condition is true. Let u and v be two vertices such that u is colored by k and v is colored by k' . If $d(u, v) = 1$, then the first or the second condition is true. If $d(u, v) = 2$ and if $k > 5$, then the first or the second condition is true. If $d(u, v) = 2$ and $k = 5$, then u is obtained by 5-3-add on vertex v and the vertex v is at distance 4 from other vertices colored by k . If $d(u, v) = 3$ and $k = 7$, then the first or the second condition is true. If $d(u, v) = 3$ and $k \leq 6$ ($k' = 6$, respectively), then u (u or v , respectively) is obtained by 6-3-add on v (u or v , respectively) and the vertex v (u or v , respectively) is at distance 4 from other vertex colored by k (k or k' , respectively). If $d(u, v) = 5$, then the S -coloring cannot be extended to the whole graph. In every case, we have u or v added by one of the operations or u and v at distance 4. Hence, we obtain that the third condition is true. \square

Theorem 5.4. Let $S = (1, 2, k, k')$, with $5 \leq k \leq 7$ and $6 \leq k' \leq 7$. A graph G is S -colorable if and only if one of these three following conditions is true:

1. $\beta(G) \leq 3$;
2. $\beta(G) = 4$ and there exist two vertices u and v in a vertex cover of size 4 such that $d(u, v) \geq 3$;
3. The graph G is obtained from a bipartite multi-graph $G' = (A, B)$ where every edge is subdivided three times and where we can do the following operations:
 - (a) if $k = 6$ and $k' = 7$, we can merge two adjacent subdivided vertices of an edge uv where u has degree 1 and u that come from B ;
 - (b) if $k = 6$ and $k' = 6$, we can take an independent set of edges from $E(G')$ and merge two adjacent subdivided vertices that come from these edges;
 - (c) if $k = 5$ and $k' = 6$, we can take a set of edges from $E(G')$ such that no edge have an extremity in common in B and merge two adjacent subdivided vertices that come from these edges;
 - (d) we can add leaves on central subdivided vertices or, if two vertices have been merges, on one of the two subdivided vertices. We can never add leaves on two vertices at distance 2. We can do 3-adds, a 2-add or add leaves on vertices that come from A or B .
 - (e) if $k \leq 6$, we can do a 6-3-add, a 6-2a-add or a 6-2b-add on vertices that come from A . If $k' \leq 6$, we can do a 6-3-add, a 6-2a-add or a 6-2b-add on vertices that come from A and B such that no 6-2b-add have been made on a neighbor of a vertex where we have added leaves;
 - (f) if $k = 5$, instead of doing a 6-3-add or a 6-2a-add on vertices that come from A , we can do a 5-3-add or several 6-3-adds and 6-2a-adds and instead of do a 6-2b-add on vertices that come from A , we can do a 5-2-add.

Proof. Let G be a graph for which one of the three conditions is true. If the first condition is true, then we color the vertices of the vertex cover by colors 2, k and k' and the remaining vertices by color 1. If the second condition is true, then we color u and v by color 2, the other vertices of the vertex cover are colored by colors k and k' and the remaining vertices are colored by 1. If the third condition is true, then we color the central subdivided vertices by color 2 (if two subdivided vertices have been merged, we color the possible vertex on which leaves have been added by color 2), the vertices that come from A by color k' , the vertices that come from B by color k , the vertices that have been added as in Figure 10 and the remaining vertices by color 1. To conclude, if one of the three conditions is true, then G is S -colorable.

Suppose that G is S -colorable. If no vertex is colored by k or k' , then the first condition is true. Let u and v be two vertices such that u is colored by k and v is colored by k' . If $d(u, v) = 1$, then the first or the second condition is true. If $d(u, v) = 2$ and if $k > 5$, then the first or the second condition is true. If $d(u, v) = 2$ and $k = 5$, then u is obtained by 5-3-add or 5-2-add on

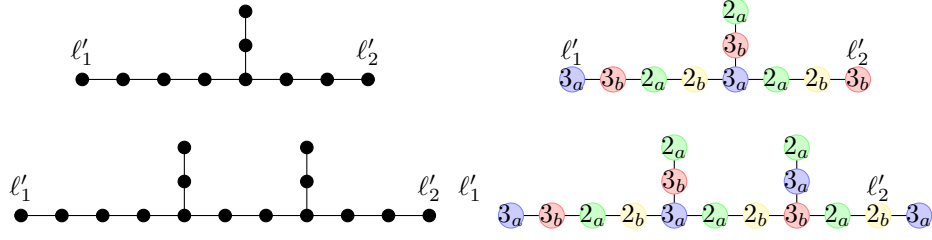


Figure 11: The graphs L' (at the bottom) and J' (at the top) and a $(2, 2, 3, 3)$ -coloring of L' and J' (on the right).

the vertex v and the vertex v is at distance 3 or 4 from other vertices colored by k . If $d(u, v) = 3$ and $k = 7$, then the first or the second condition is true. If $d(u, v) = 3$ and $k \leq 6$ ($k' = 6$, respectively), then u (u or v , respectively) is obtained by 6-3-add, by 6-2a-add or by 6-2b-add on v (u or v , respectively) and the vertex v (u or v , respectively) is at distance 4 or 3 from the other vertices colored by k (k or k' , respectively). If $d(u, v) = 5$, then the S -coloring cannot be extended to the whole graph. In every case, we have u or v added by one of the operations or u and v at distance 3 or 4. Thus, we obtain that the third condition is true. \square

In the following proof, we use the construction from Lemma 3.2. The existence properties of a S -coloring with the properties of Lemma 3.2 will not be verified anymore. However, using the different $(2, 2, 3, 3)$ -colorings from the proof, the reader can be convinced that there exist a S -coloring of the whole constructed graph.

Theorem 5.5. *The problem $(2, 2, 3, 3)$ -COL is NP-complete.*

Proof. Let L' and J' be subcubic graphs from Figure 11 with connectors the vertices ℓ'_1 and ℓ'_2 . Note that a vertex of degree at least 3 can only be colored with colors 3_a or 3_b . In the constructed graph (Lemma 3.2), a vertex of degree at least 3 will be in an induced Y_2 and there does not exist a S -coloring of Y_2 with the central vertex colored by colors 2_a or 2_b . For the same reason, every connectors should be colored by colors 3_a or 3_b (these vertices will be in an induced Y_2 in the constructed graphs). If two vertices u and v are of degree at least 3 or are connectors and $d(u, v) = 4$, then u and v can only have the same color 3, the vertices in the path between u and v cannot be colored anymore if u and v have the different colors.

If two connectors of L' have the same color, then the vertices in the path of length 4 between the central vertex of degree 3 and a connector cannot be colored. Thus, in every $(2, 2, 3, 3)$ -coloring of L' , the connectors should have the same color 3. A $(2, 2, 3, 3)$ -coloring of L' is illustrated in Figure 11.

If two connectors of J' have different color 3, then the vertices in the path of length 4 between one of the central vertex of degree 3 and a connector cannot

$S' = (2, 2, 3, 4)$, $S' = (1, 4, 4, 4)$ or $S' = (1, 2, 5, 6)$, and NP-complete otherwise.

Proof. Using Propositions 4.10, 5.1, 5.2, 5.3 and 5.4, we obtain that S -COL is polynomial time solvable in the given cases. Using Propositions 4.12, 4.14 and 4.16, we obtain that S -COL is NP-complete in the other cases. \square

Conclusion

We finish with proposing two conjectures about S -COL, with Conjecture 2 implying Conjecture 1.

Conjecture 1. *There exists a dichotomy between NP-complete problems and polynomial time solvable problems among the instances of S -COL, where S is fixed.*

Conjecture 2. *Let S be a non decreasing list of integers. At least one condition is true about S -COL:*

1. *Every S -colorable graph can be obtained from a bipartite graph using some operations;*
2. *Every S -colorable graph has bounded treewidth;*
3. *The problem S -COL is NP-complete.*

References

- [1] G. Argirosso. Polynomial instances of the packing coloring problem. *Electronic Notes in Discrete Mathematics*, pages 363–368, 2011.
- [2] G. Argirosso, G. Nasini, and P. Torres. The packing coloring problem for $(q, q-4)$ graphs. *Lecture Notes in Computer Science*, 7422:309–319, 2012.
- [3] D. Bienstock, N. Robertson, P. Seymour, and R. Thomas. Quickly excluding a forest. *Journal of Combinatorial Theory*, 52:274–283, 1991.
- [4] B. Bresar, S. Klavzar, and D. F. Rall. On the packing chromatic number of cartesian products, hexagonal lattice, and trees. *Discrete Applied Mathematics*, 155:2303 – 2311, 2007.
- [5] B. Courcelle. The monadic second-order logic of graphs i. recognizable sets of finite graphs. *Information and Computation*, 85:12–75, 1990.
- [6] N. G. de Bruijn and P. Erdős. A colour problem for infinite graphs and a problem in the theory of relations. *Nederl. Akad. Wetensch. Proc. Ser. A*, 54:371–373, 1951.
- [7] J. Ekstein, J. Fiala, P. Holub, and B. Lidický. The packing chromatic number of the square lattice is at least 12. *arXiv:1003.2291v1*, 2010.

- [8] J. Fiala and P. A. Golovach. Complexity of the packing coloring problem for trees. *Discrete Applied Mathematics*, 158:771–778, 2010.
- [9] A. S. Finbow and D. F. Rall. On the packing chromatic number of some lattices. *Discrete Applied Mathematics*, 158:1224–1228, 2010.
- [10] N. Gastineau. Complexity of the (s_1, s_2, \dots, s_k) -packing coloring problem. in *Bordeaux Graph Workshop*, Conference, 2012.
- [11] N. Gastineau, H. Kheddouci, and O. Togni. Subdivision into i -packings and S -packing chromatic number of some lattices. *Arxiv*, 2013.
- [12] W. Goddard, S. M. Hedetniemi, S. T. Hedetniemi, J. M. Harris, and D. F. Rall. Broadcast chromatic numbers of graphs. *Ars Combinatoria*, 86:33–49, 2008.
- [13] W. Goddard and H. Xu. The S -packing chromatic number of a graph. *Discussiones Mathematicae Graph Theory*, 32:795–806, 2012.
- [14] W. Goddard and H. Xu. A note on S -packing colorings of lattices. *Discrete Applied Mathematics*, 166:255–262, 2014.
- [15] A. J. Hoffman and R. R. Singleton. On Moore graphs with diameters 2 and 3. *IBM Journal of Research and Development*, 5:497–504, 1960.
- [16] N. Robertson and P. Seymour. Graph minors iii: Planar tree-width. *Journal of Combinatorial Theory*, 36:49–64, 1984.
- [17] S. Shaebani. On the b -chromatic number of regular graphs without 4-cycle. *Discrete Applied Mathematics*, 160:1610–1614, 2012.
- [18] A. Sharp. Distance coloring. In *ESA*, pages 510–521, 2007.
- [19] K. Wagner. Über eine eigenschaft der ebenen komplexe. *Math. Ann.*, 114:570–590, 1937.