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S-Packing Colorings of Cubic Graphs

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Given a non-decreasing sequence $S = (s_1, s_2, \ldots, s_k)$ of positive integers, an *S-packing coloring* of a graph G is a mapping c from $V(G)$ to $\{s_1, s_2, \ldots, s_k\}$ such that any two vertices with the ith color are at mutual distance greater than s_i, $1 \leq i \leq k$. This paper studies S-packing colorings of (sub)cubic graphs. We prove that subcubic graphs are $(1, 2, 2, 2, 2, 2, 2)$-packing colorable and $(1, 1, 2, 2, 2)$-packing colorable. For subdivisions of subcubic graphs we derive sharper bounds, and we provide an example of a cubic graph of order 38 which is not $(1, 2, \ldots, 12)$-packing colorable.

Keywords: graph, coloring, packing chromatic number, cubic graph.

1 Introduction

A proper coloring of a graph G is a mapping which associates a color (integer) to each vertex such that adjacent vertices get distinct colors. In such a coloring, the color classes are stable sets (1-packings). As an extension, a d-distance coloring of G is a proper coloring of the d-th power G^d of G, i.e. a partition of $V(G)$ into d-packings (sets of vertices at pairwise distance greater than d). While Brook's theorem implies that all cubic graphs except the complete graph K_4 of order 4 are properly 3-colorable, many authors studied 2-distance colorings of cubic graphs.

The aim of this paper is to study a mixing of these two types of colorings, i.e. colorings of (sub)cubic graphs in which some colors classes are 1-packings while other are d-packings, $d \geq 2$. Such colorings can be expressed using the notion of S-packing coloring. For a non-decreasing sequence $S = (s_1, s_2, \ldots, s_k)$ of positive integers, an *S-packing coloring* (or simply S-coloring) of a graph G is a coloring of its vertices with colors from $\{s_1, s_2, \ldots, s_k\}$ such that any two vertices with the ith color are at mutual distance greater than s_i, $1 \leq i \leq k$. The color class of each color s_i is thus an s_i-packing. The graph G is *S-colorable* if there exists an S-coloring and it is *S-chromatic* if it is S-colorable but not S'-colorable for any

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$S' = (s_1, s_2, \ldots, s_j)$ with $j < k$ (notice that Goddard et al. [14] define differently the S-chromaticness for infinite graphs).

A $(d, \ldots, d)$-coloring is thus a d-distance k-coloring, where k is the number of d (see [17] for a survey of results on this invariant) while a $(1, 2, \ldots, d)$-coloring is a packing coloring. The packing chromatic number $\chi_\rho(G)$ of G is the integer k for which G is $(1, \ldots, k)$-chromatic. This parameter was introduced by Goddard et al. [12] under the name of *broadcast chromatic number* and the authors showed that deciding whether $\chi_\rho(G) \leq 4$ is NP-hard. A series of works [3, 6, 8, 9, 12, 18] considered the packing chromatic number of infinite grids. For sequences S other than $(1, 2, ..., k)$, S-packing colorings were considered more recently [11, 13, 14]. Other papers are about the complexity class of the decision problem associated to the S-packing coloring problem [7, 10].

Regarding subcubic graphs, the packing chromatic number of the hexagonal lattice and of the infinite 3-regular tree is 7 and at most 7, respectively. Recently, Brešar et al. [4], have proven that the packing chromatic number of some cubic graphs, namely the base-3 Sierpiński graphs, is bounded by 9. Goddard et al. [12] asked what is the maximum of the packing chromatic number of a cubic graph of order n. For 2-distance coloring of cubic graphs, Cranston and Kim have recently shown [5] that any subcubic graph is $(2, 2, 2, 2, 2, 2, 2, 2)$-colorable (they in fact proved a stronger statement for list coloring). For planar subcubic graphs G, there are also sharper results depending on the girth of G [2, 5, 16].

In this paper, we study S-packing colorings of subcubic graphs for various sequences S starting with one or two '1'. We also compute the distribution of S-chromatic cubic graphs up to 20 vertices, for three sequences S. The corresponding results are reported on Tables 1, 2, and 3. They are obtained by an exhaustive search, using the lists of cubic graphs maintained by Gordon Royle [15]. The paper is organized as follows: Section 2 is devoted to the study of $(1, k, \ldots, k)$-colorings of subcubic graphs for $k = 2$ or 3; Section 3 to $(1, 1, 2, \ldots)$-colorings; Section 4 to $(1, 2, 3, \ldots)$-colorings and Section 5 concludes the paper by listing some open problems.

1.1 Notation

To describe an S-coloring, if an integer s is repeated in the sequence S, then we will denote the colors s by $s_a, s_b, \ldots$.

The *subdivided graph* $S(G)$ of a (multi)graph G is the graph obtained from G by subdividing each edge once, i.e. replacing each edge by a path of length two. In $S(G)$, vertices of G are called *original* vertices and other vertices are called *subdivision* vertices. Let us call a graph d-*irregular* if it has no adjacent vertices of degree d. Notice that graphs obtained from subcubic graphs by subdividing each edge at least once are 3-irregular graphs.

The following method (that is inspired from that of Cranston and Kim [5]) is used in the remainder of the paper to produce a desired coloring of a subcubic graph (except for Theorem 3): for a graph G and an edge $e = xy \in E(G)$, a *level ordering* of (G, e) is a partition of $V(G)$ into levels $L_i = \{v \in V(G) : d(v, e) = i\}$, $0 \leq i \leq \epsilon(e)$, with $\epsilon(e) = \max(\{d(u, e), u \in V(G)\}) \leq diam(G)$. The vertices are then colored one by one, from level $\epsilon(e)$ to 1, while preserving some properties. These properties are used at the end to allow to color the vertices x and y by recoloring possibly some vertices in their neighborhoods.

Two vertices u and v of G are called *siblings* if they are not adjacent, are on the same level L_i for some $i \geq 1$ and have a common neighbor in L_{i-1}. A vertex in a level i is a k-*vertex* if it has k neighbors in $L_i \cup L_{i+1}$. Notice that a 2-vertex in a subcubic graph has at most one

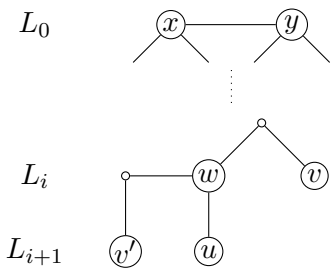


Figure 1: Level ordering of a subcubic graph from the edge xy. The vertices v, w are siblings but u, v' are not siblings.

sibling (see Figure 1). Given a (partial) coloring c of G, let $C_1(u) = \{c(v) : uv \in E(G)\}$ and $C_2(u) = \{c(v) : d(u,v) = 2, \text{ with } u, v \text{ not siblings}\}$.

2 $(1, k, \ldots, k)$-coloring

In this section, $(1, k, \ldots, k)$-colorings of subcubic graphs are studied for $k = 2$ or 3.

2.1 $(1, 3, \ldots, 3)$-coloring

The following proposition is used to obtain an S-coloring of a subdivided graph:

Proposition 1. *Let G be a graph and $S = (s_1, \ldots, s_k)$ be a non-decreasing sequence of integers. If G is S-colorable then $S(G)$ is $(1, 2s_1 + 1, \ldots, 2s_k + 1)$-colorable.*

Proof. Let c be an S-coloring of G. Every pair of vertices $u, v \in V(G)$ such that $d(u,v) = d$ become at distance $2d$ in $S(G)$. Therefore, every set of vertices in $V(G)$ forming an i-packing also forms a $(2i+1)$-packing in $S(G)$. Using color 1 on subdivision vertices and using the coloring c (considering the sequence differently) on original vertices, we obtain a $(1, 2s_1 + 1, \ldots, 2s_k + 1)$-coloring of $S(G)$. □

Corollary 1. *For every subcubic graph G, $S(G)$ is $(1, 3, 3, 3)$-colorable.*

Proof. Brooks' theorem asserts that every subcubic graph except K_4 is $(1, 1, 1)$-colorable. Hence, by Proposition 1, every subcubic graph G except K_4 is such that $S(G)$ is $(1, 3, 3, 3)$-colorable. We define a $(1, 3, 3, 3)$-coloring of $S(K_4)$ as follows: let $\gamma : E(K_4) \to \{a, b, c\}$ be a proper edge 3-coloring of K_4. Put color 1 on all four original vertices of K_4 and put color $3_{\gamma(e)}$ on each subdivision vertex corresponding to edge e of K_4. □

Goddard et al. [12] characterized $(1, 3, 3)$-colorable graphs as the graphs obtained from any bipartite multigraph by subdividing it and adding leaves on original vertices. Therefore, there are many subdivided subcubic graphs that are not $(1, 3, 3)$-colorable (for instance $S(C_3) = C_6$), showing that the bound of Corollary 1 is tight in a certain sense.

2.2 $(1, 2, \ldots, 2)$-coloring

Notice that a vertex of degree at least 3 in a $(1, 2, 2)$-colored graph can not be colored by 1. Thus, a $(1, 2, 2)$-colorable graph does not contain three vertices of degree larger than 2 at mutual distance at most 2 and in particular no cubic graph is $(1, 2, 2)$-colorable. However, there exist $(1, 2, 2)$-colorable subcubic graphs and it has been recently proved [10] that determining if a subcubic bipartite graph is $(1, 2, 2)$-colorable is NP-complete.

Theorem 1. *Every subcubic graph is* $(1, 2, 2, 2, 2, 2, 2)$*-colorable.*

Proof. Let G be a subcubic graph and let $e = xy$ be any edge of G. Define a level ordering L_i, $0 \leq i \leq r = \epsilon(e)$, of (G, e).

We first construct a coloring c of the vertices of G from level r to 1 and with colors from the set $C = \{1, 2_a, 2_b, 2_c, 2_d, 2_e, 2_f\}$, that satisfies the following properties:

i) color 1 is used as often as possible, i.e. when coloring a vertex u, if no neighbor is colored 1, then u is colored 1;

ii) if u is colored by 2, then there is a subsidiary color $\tilde{c}(u) \in C$ different from $c(u)$ such that $\tilde{c}(u) \notin C_1(u) \cup C_2(u)$, but with possibly $\tilde{c}(u) = c(v)$ if u and v are siblings.

The set L_r induces a disjoint union of paths and cycles in G. Since paths and cycles are $(1, 2, 2, 2)$-colorable, we are able to construct a coloring of the vertices of L_r as follows. Start by coloring each path/cycle with colors $\{1, 2_a, 2_b, 2_c\}$ with color 1 used as often as possible. We suppose that color 1 is used for the end-vertices (vertices of degree 1) of the paths except in the case the path has length 1. Remark that a vertex of a cycle of L_r or a vertex of degree 2 inside a path of L_r can be at distance 2 of at most one vertex in L_r outside this cycle/path. A vertex of a path of length 1 of L_r can be at distance 2 of at most two vertices in L_r outside this path. We then recolor each vertex u of a path of length 1 in L_r with color from $\{2_a, 2_b, 2_c\}$ by a color from $\{2_d, 2_e, 2_f\}$ not given to vertices at distance at most 2 from u and we set $\tilde{c}(u) = 2_a$. Then, for each pair of vertices u, v in different paths/cycles at distance 2 both colored by 2_a (2_b or 2_c respectively), set $c(u) = 2_d$ (2_e or 2_f, respectively). Afterwards, for every vertex u of color 2_a (2_b, 2_c, 2_d, 2_e or 2_f, respectively), set $\tilde{c}(u) = 2_d$ (2_e, 2_f, 2_a, 2_b or 2_c, respectively). Then, the produced coloring is a partial $(1, 2, 2, 2, 2, 2, 2)$-coloring of G and Property ii) is satisfied.

Assume that we have already colored all vertices of G of levels from r to $i + 1$ and that we are going to color vertex $u \in L_i$, $1 \leq i \leq r - 1$. If $1 \notin C_1(u)$ then set $c(u) = 1$ (Property i) is then satisfied). If u is a 0-vertex, then it has been colored by 1. If u is a 1-vertex, then $|C_1(u) \cup C_2(u)| \leq 3$ and u has at most two siblings v and v'. In this case, we can set a color and a subsidiary color to u from the set $C \setminus (C_1(u) \cup C_2(u) \cup \{c(v), c(v')\})$. Therefore, we now suppose that u is a 2-vertex.

If $1 \in C_1(u)$, then let u_1 be the neighbor of u of color 1 and let u_2 be the other neighbor of u, if any. By construction, either $c(u_2) = 1$ or $1 \in C_1(u_2)$, hence $|C_1(u) \cup C_2(u)| \leq 5$. In that case there are at least two colors $\{2_\alpha, 2_\beta\} \subset C \setminus \{C_1(u) \cup C_2(u)\}$ for some $\alpha, \beta \in \{a, \ldots, f\}$, with possibly, if u has a sibling v, $2_\beta = c(v)$. Then set $c(u) = 2_\alpha$ and $\tilde{c}(u) = 2_\beta$ (Property ii) is then satisfied). Figure 2 illustrates this case.

Finally, it remains to color vertices of L_0, i.e., x and y. If $1 \in C_1(x) \cap C_1(y)$ then, by Property i), the neighbor x_2 of x colored by 2, if any, has a neighbor of color 1 and the same goes for y, with y_2 being the neighbor of y colored by 2, if any. Hence $|C_1(x) \cup C_2(x)| \leq 5$,

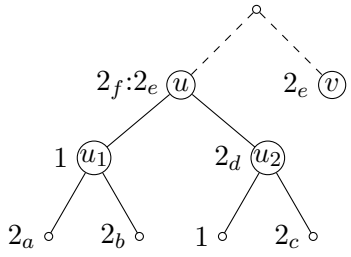


Figure 2: A configuration in the proof of Theorem 1, when coloring vertex u. The label 2_f:2_e on u means that $c(u) = 2_f$ and $\tilde{c}(u) = 2_e$.

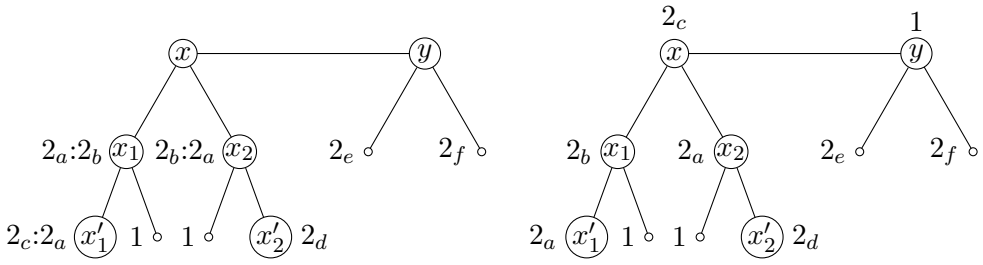


Figure 3: A configuration in the proof of Theorem 1, before (on the left) and after (on the right) coloring x and y.

$|C_1(y) \cup C_2(y)| \leq 5$ and there remains at least two colors 2 available for x and two colors 2 available for y. Therefore x and y can be assigned a different color 2.

If $1 \in C_1(x)$ but $1 \notin C_1(y)$ (or $1 \in C_1(y)$ but $1 \notin C_1(x)$, by symmetry), then set $c(y) = 1$. We recall that x_2 is the neighbor of x not colored by 1. If $C_1(x) \cup C_2(x) = C$, then set $c(x) = c(x_2)$ and $c(x_2) = \tilde{c}(x_2)$, else give to x an available color.

Otherwise, $1 \notin C_1(x) \cup C_1(y)$. Then set $c(y) = 1$ and we show that there is always a color 2 to assign to x. If $|C_1(x) \cup C_2(x)| \leq 6$, then there is a color available for x. Else, let x_1, x_2 be the two neighbors of x other than y and let x'_1 (x'_2, respectively) be the neighbor of x_1 (x_2, respectively) colored 2 other than x (no more than one, as x_1 and x_2 both have a neighbor colored by 1). Suppose, without loss of generality, that $c(x_1) = 2_a$, $c(x_2) = 2_b$, $c(x'_1) = 2_c$ and $c(x'_2) = 2_d$. If $\tilde{c}(x_1) \in \{2_d, 2_e, 2_f\}$ then recolor x_1 by its subsidiary color $\tilde{c}(x_1)$ and set $c(x) = 2_a$. Similarly, if $\tilde{c}(x_2) \in \{2_c, 2_e, 2_f\}$ then recolor x_2 by its subsidiary color $\tilde{c}(x_2)$ and set $c(x) = 2_b$. Else, $\tilde{c}(x_1) = 2_b$ and $\tilde{c}(x_2) = 2_a$. Recolor x'_1 by its subsidiary color $\tilde{c}(x'_1)$ and set $c(x) = 2_c$. If $\tilde{c}(x'_1) = 2_a = c(x_1)$, then switch the colors of x_1 and x_2 (this is possible since $\tilde{c}(x_1) = c(x_2)$ and $\tilde{c}(x_2) = c(x_1)$). Figure 3 illustrates this case.

Therefore, we obtain, in all cases, a $(1, 2, 2, 2, 2, 2, 2)$-coloring of G. □

The Petersen graph is an example of cubic graph which is not $(1, 2, 2, 2, 2, 2)$-colorable, showing that the result of Theorem 1 is tight in a certain sense. However, the experiments reported on Table 1 suggest that the Petersen graph could be the only non $(1, 2, 2, 2, 2, 2)$-colorable subcubic graph. These experiments have been done using a computer to check every possible color configuration and by verifying that it is a S-coloring, for sequences S

with $s_1 = 1$, $|S| \geq 4$ and $s_i = 2$, for $i \geq 2$. We have considered the list of cubic graphs from Gordon Royle [15] .

$n \backslash S$	$(1,2,2,2)$	$(1,2,2,2,2)$	$(1,2,2,2,2,2)$	$(1,2,2,2,2,2,2)$
4	1	0	0	0
6	1	1	0	0
8	2	1	2	0
10	11	7	0	1
12	11	74	0	0
14	254	250	5	0
16	1031	3017	12	0
18	15960	25297	44	0
20	178193	332045	251	0
22	2481669	4835964	1814	0

Table 1: Number of S-chromatic cubic graphs of order n up to 22.

Furthermore, as the following proposition shows, even some bipartite cubic graphs are not $(1,2,2,2,2,3)$-colorable.

Proposition 2. *There exist bipartite cubic graphs that are not* $(1,2,2,2,2,3)$*-colorable.*

Proof. The cubic graph depicted in Figure 4 is bipartite and is $(1,2,2,2,2,2)$-colorable, as shown on the figure. Let (A,B) be the two subsets of vertices that form a bipartition of this graph. Suppose this graph is $(1,2,2,2,2,3)$-colorable and let c be a $(1,2,2,2,2,3)$-coloring and X_1 be the set of vertices colored 1. Remark that the cardinality of any 2-packing is at most 2 and that any pair of vertices (u,v) included in A or in B is such that $d(u,v) \leq 2$. We have $|X_1| \geq 5$, as at most one vertex can be colored by 3 (since the diameter of the graph is 3) and at most two vertices can be colored the same color 2.

First, if $X_1 \subseteq A$ or $X_1 \subseteq B$, then each remaining vertex should be colored differently in the other partition, which is impossible since $|A| = |B| = 7$.

Second, if there are vertices colored by 1 in A and B, then the only possibility in order to have $|X_1| \geq 5$ is to have one vertex colored by 1 in one partition and four vertices colored 1 in the other partition (since othewise we obtain adjacent vertices of color 1). Suppose, without loss of generality, that $|X_1 \cap A| = 1$ and $|X_1 \cap B| = 4$. Exactly three vertices are not colored 1 in B. Consequently, only three pairs of vertices can have the same color 2 and the three vertices not colored by 1 in A cannot be all colored with the remaining colors 2 and 3. □

The next results show that there are sub-families of subcubic graphs that can be colored with fewer colors.

Theorem 2. *Every 3-irregular subcubic graph is* $(1,2,2,2)$*-colorable.*

Proof. Let G be a 3-irregular graph and let $e = xy$ be any edge of G such that x and y are both of degree at most 2. If no such edge exists, then the graph is the subdivision $S(H)$ of some subcubic graph H where leaves could be added on original vertices of degree 2 and thus G is $(1,3,3,3)$-colorable by Corollary 1. Define a level ordering L_i, $0 \leq i \leq r = \epsilon(e)$, of (G,e).

We construct a coloring c of the vertices of G from level r to 1 and with colors from the set $\{1, 2_a, 2_b, 2_c\}$, that satisfies the following properties:

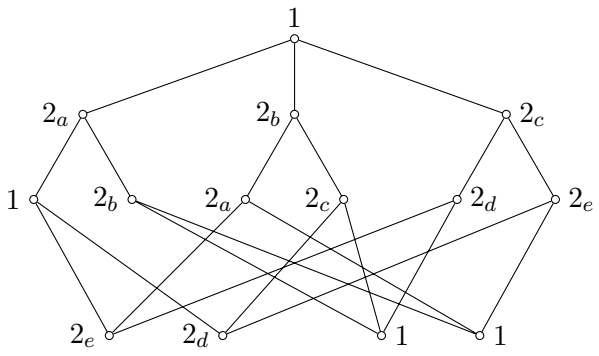


Figure 4: A cubic bipartite $(1, 2, 2, 2, 2, 2)$-chromatic graph of order 14.

i) color 1 is used as often as possible for vertices of degree at most 2, i.e. when coloring a vertex u of degree at most 2, if no neighbor is colored by 1, then u is colored by 1;

ii) every vertex of degree 2 is colored by 1 when first coloring vertices of L_i, except if the connected component containing this vertex in L_i is a path of order 2 (in which case one of the two vertices is colored 1).

The set L_r induces a disjoint union of paths of order at most 3 in G. Since paths are $(1, 2, 2)$-colorable, the vertices of L_r can be $(1, 2, 2, 2)$-colored. Moreover, in every path of order 3 in L_r, the central vertex has degree 3, thus a color 1 could be given to every vertex of degree 2. If the path is of order 2, one of its end-vertices is colored by 1. Thus, Properties i) and ii) are satisfied.

Assume that we have already colored all vertices of G of levels from r to $i + 1$ and that we are going to color vertex $u \in L_i$, $1 \leq i \leq r - 1$. We consider two cases depending on the degree of u:

Case 1. u is of degree 3.

If $1 \notin C_1(u)$, then u can be colored by 1. Let u_1 and u_2 be the colored neighbors of u, with $c(u_1) = 1$. By Property i), either u_2 or a colored neighbor of u_2 has color 1. Hence, we have $|C_1(u) \cup C_2(u)| \leq 3$ and u can be colored some color 2.

Case 2. u is of degree 1 or 2.

If u has degree 1, then we can color u by 1. If $1 \notin C_1(u)$, then we can set $c(u) = 1$. Otherwise, let u_1 be the colored neighbor of u, if any. If u_1 is of degree 3, let $u_{1,1}$ and $u_{1,2}$ be the colored neighbor of u, let $u_{1,1,1}$ be the neighbor of $u_{1,1}$ different from u and let $u_{1,2,1}$ be the neighbor of $u_{1,2}$ different from u. Since $1 \in C_1(u)$, $c(u_1) = 1$ and thus $c(u_{1,1}) \neq 1$ and $c(u_{1,2}) \neq 1$. Therefore, by Property i), $c(u_{1,1,1}) = c(u_{1,2,1}) = 1$. Thus, u_1 can be recolored by some color 2 and we can set $c(u) = 1$. If u_1 is of degree at most 2, then, since $|C_1(u_1) \cup C_2(u_1)| \leq 3$, we can recolor u_1 by a color 2. Thus, we can set $c(u) = 1$.

Finally, it remains to color vertices of L_0, i.e. x and y. Let x_1 be the possible neighbor of x different from y and let y_1 be the possible neighbor of y different from x. We consider three cases that cover all the possibilities by symmetry:

Case 1. x_1 and y_1 both have degree 3.

If their neighbors different from x and y are not adjacent between them, then, by Property ii), these vertices have color 1 (they have not been recolored when coloring the vertices of L_1

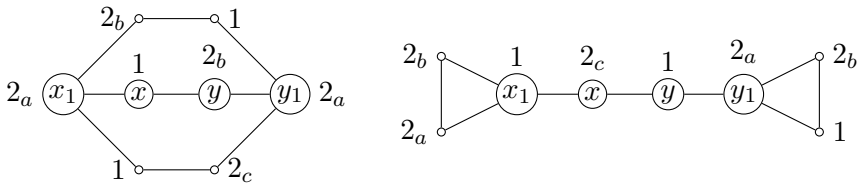


Figure 5: The graphs $G_{(1,2,2,2)}$ (on the left) and $G'_{(1,2,2,2)}$ (on the right) from Proposition 2.

since x_1 and y_1 have degree 3) and x_1 and y_1 have some color 2. Thus we can set $c(x) = 1$ and some color 2 to y, as $|C_1(y) \cup C_2(y)| \leq 3$. Suppose that the two vertices of $N(x_1) \setminus \{x\}$ are each adjacent to a different vertex of $N(y_1) \setminus \{y\}$. Thus, G is the graph $G_{(1,2,2,2)}$ from Figure 5 and is $(1,2,2,2)$-colorable. Suppose that the two vertices of $N(x_1) \setminus \{x\}$ are adjacent and that the two vertices of $N(y_1) \setminus \{y\}$ are adjacent. In this case, G is the graph $G'_{(1,2,2,2)}$ from Figure 5 and is $(1,2,2,2)$-colorable. Suppose that only one vertex of $N(x_1) \setminus \{x\}$ is adjacent to a vertex of $N(y_1) \setminus \{y\}$. Let $x_{1,1}$ and $y_{1,1}$ be these two adjacent neighbors, the other neighbors are colored by 1 by Property ii). One of these two vertices is colored by 1 and the other one is colored by 2. Suppose without loss of generality that $c(x_{1,1}) = 1$. Hence, we have $|C_1(x) \cup C_2(x)| \leq 3$ and we can color x by a color 2 and set $c(y) = 1$. Suppose now that the two vertices of $N(x_1) \setminus \{x\}$ are adjacent and that the two vertices of $N(y_1) \setminus \{y\}$ are not adjacent. By Property ii), the neighbors of y_1 have color 1 and y_1 has some color 2. In this case, we color y by 1, x by another color 2, x_1 by 1 and the two adjacent neighbors of x_1 by colors from $C \setminus \{1, c(x)\}$.

Case 2. x_1 has degree at most 2 and y_1 has degree 3.

Since $|C_1(y_1) \cup C_2(y_1)| \leq 3$, then y_1 can be recolored by some color 2. If x_1 and y_1 are not adjacent, then x_1 is colored by 1 by Property ii). Otherwise (x_1 and y_1 are adjacent), we can color x_1 by 1 since y_1 has not been colored by 1. Afterward, we set $c(y) = 1$, and since $|C_1(x) \cup C_2(x)| \leq 3$ we can set a color 2 to x.

Case 3. x_1 and y_1 are both of degree at most 2.

If x_1 and y_1 are adjacent, then the graph is C_4 which is trivially $(1,2,2,2)$-colorable. If x_1 and y_1 are not adjacent, then, by Property ii), they both have color 1, $|C_1(x) \cup C_2(x)| \leq 2$ and $|C_1(y) \cup C_2(y)| \leq 2$. Thus, we can set some colors 2 to x and y.

Therefore, we obtain in all cases a $(1,2,2,2)$-coloring of G. □

Remark that the 5-cycle C_5 is 3-irregular and is not $(1,2,2)$-colorable, hence the result of Theorem 2 is tight in a certain sense. However, there are 3-irregular subcubic graphs that are $(1,2,2,3)$-colorable. The graphs from Figure 5 are such examples (the color 2_c can be replaced by color 3).

We end this section with some results on subdivided graphs. Let $\delta(G)$ be the minimum degree of G. In the following theorem, we suppose that $\delta(G) \geq 3$ in order to avoid cycles with few vertices of degree at least 3 (every cycle, except C_5, is $(1,2,2)$-colorable).

Proposition 3. *For every graph G with $\delta(G) \geq 3$, if $S(G)$ is $(1,2,2)$-colorable, then G is bipartite.*

Proof. Suppose $S(G)$ is $(1,2,2)$-colorable and G contains an odd cycle. In every $(1,2,2)$-coloring of a graph, every vertex of degree at least 3 should be colored some color 2 (if a

vertex of degree at least 3 is colored by 1, the coloring cannot be extended to the neighbors of this vertex). Therefore, if G contains an odd cycle, then $S(G)$ contains a cycle with an odd number of vertices of degree 3 and the colors 2_a and 2_b are not sufficient to alternately color these vertices. Hence $S(G)$ is not $(1,2,2)$-colorable. □

Since every bipartite graph G is $(1,1)$-colorable, by Proposition 1, $S(G)$ is $(1,3,3)$-colorable (and also $(1,2,2)$-colorable and $(1,2,3)$-colorable). Thus, we obtain the following corollary.

Corollary 2. *For every graph G with $\delta(G) \geq 3$,*

$S(G)$ is $(1,2,2)$-colorable $\Leftrightarrow$ $S(G)$ is $(1,2,3)$-colorable $\Leftrightarrow$ $S(G)$ is $(1,3,3)$-colorable $\Leftrightarrow$ G is bipartite.

3 $(1,1,2,\ldots)$-coloring

Similarly with $(1,2,2)$-coloring, it has been recently proved [10] that determining if a subcubic or a cubic graph is $(1,1,2)$-colorable is NP-complete. Remind that bipartite graphs are $(1,1)$-colorable. For non-bipartite subcubic graphs, we prove the following result using a different argument than for the previous theorems. Using the method of proofs of Theorems 1 and 2 allows to show that subcubic graph are $(1,1,2,2,2,2)$-colorable.

Theorem 3. *Every subcubic graph is $(1,1,2,2,2)$-colorable.*

Proof. Let G be a subcubic graph. For a vertex u of $V(G)$, we denote by $B_2(u)$ the set $\{v \in V(G) |\ d_G(u,v) \leq 2,\ u \neq v\}$. A set $X \subseteq V(G)$ is an *odd-cut set* if the graph induced by $V(G) \setminus X$ is a bipartite graph. Such set can be obtained by removing one vertex per odd cycle from $V(G)$. Let G_X be the graph with vertex set X and edge set $\{uv |\ u,\ v \in X,\ d_G(u,v) \leq 2,\ u \neq v\}$. The proof consists of proving the existence of an odd-cut set X such that G_X is a subcubic graph (not necessarily connected) and G_X has no connected component isomorphic to K_4.

If G_X is subcubic and has no connected component isomorphic to K_4, then we can construct a coloring c of G with colors from the set $C = \{1_a, 1_b, 2_a, 2_b, 2_c\}$ as follows. Since the graph induced by $V(G) \setminus X$ is bipartite, we can color the vertices of $V(G) \setminus X$ with colors 1_a and 1_b. By Brook's Theorem, if G_X is subcubic and has no connected component isomorphic to K_4, then there exists a proper vertex-coloring c' of G_X with colors from the set $\{a,b,c\}$. For a vertex $u \in X$, we define c by $c(u) = 2_{c'(u)}$.

For a cycle C of G, we denote by $V(C)$ the set of vertices in C. For an odd-cut set $X \subseteq V(G)$ and a vertex $u \in X$, we denote by C_u a cycle such that $X \cap V(C_u) = \{u\}$. In the rest of the proof, when G_X is supposed to have an induced subgraph isomorphic to K_4, we denote by u_1, u_2, u_3 and u_4 the vertices of this induced subgraph and by U the set $\{u_1,u_2,u_3,u_4\}$. For two vertices u_i and u_j, $1 \leq i < j \leq 4$, we denote by u_{ij} a common neighbor of u_i and u_j in $V(G)$ when $d_G(u_i,u_j) = 2$. When u_{ij} exists, we denote by u'_{ij}, the possible vertex in $N(u_{ij}) \setminus \{u_i,u_j\}$.

Claim 1. For an odd-cut set X in G with minimum cardinality, the following properties hold:

(A1) for every vertex $u \in X$, there exists at least one cycle C such that $V(C) \cap X = \{u\}$ and consequently $|N(u) \cap X| \leq 1$;

(A2) for every two vertices $u, v \in X$ such that $uv \in E(G)$, we have $d_{G_X}(u) \leq 3$ and $d_{G_X}(v) \leq 3$;

(A3) for every vertex $u \in X$, $d_{G_X}(u) \leq 4$;

(A4) if there does not exist three vertices $u, v, w \in X$ such that $N(u) \cap N(v) \cap N(w) \neq \emptyset$ and $\max(d_{G_X}(u), d_{G_X}(v), d_{G_X}(w)) \geq 3$, then for every vertex $x \in X$, $d_{G_X}(x) \leq 3$.

Proof. The left part of Figure 6 illustrates a vertex $u \in X$ satisfying $d_{G_X}(u) = 4$.

(A1): If such cycle did not exist, then $X \setminus \{u\}$ would be also an odd-cut set, contradicting its minimality. Moreover, since u has two neighbors in $V(C_u)$, $|N(u) \cap X| \leq 1$.

(A2): Let u and v be two adjacent vertices of X. By Property (A1), there exist two pairs of distinct vertices $\{u_1, u_2\}$ and $\{v_1, v_2\}$ such that $\{u_1, u_2\} \subseteq N(u) \cap V(C_u)$ and $\{v_1, v_2\} \subseteq N(v) \cap V(C_v)$. Notice that both u_1 and u_2 have at least two neighbors in $V(C_u)$. Then, $|B_2(u) \setminus (V(C_u) \cup V(C_v))| \leq 3$. Since the same property holds for C_v, we have $d_{G_X}(u) \leq 3$ and $d_{G_X}(v) \leq 3$.

(A3) and (A4): Let u be a vertex of X. By Property (A1), there exist two distinct vertices u_1 and u_2 such that $\{u_1, u_2\} \subseteq N(u) \cap V(C_u)$. As previously, u_1 and u_2 have at least two neighbors in $V(C_u)$. Let u_3 be the possible vertex in $N(u) \setminus V(C_u)$. By Property (A2), if $u_3 \in X$, then $d_{G_X}(u) \leq 3$. Thus, we can now suppose that $u_3 \notin X$. Since $|(N(u_3) \setminus \{u\}) \cap X| \leq 2$, we have $|B_2(u) \cap X| \leq 4$ and consequently $d_{G_X}(u) \leq 4$. Moreover, if there exists a vertex u such that $d_{G_X}(u) = 4$, then we have $|(N(u_3) \setminus \{u\}) \cap X| = 2$ and u_3 is a common neighbor of three distinct vertices of X. Consequently, the contrapositive of Property (A4) follows.

□

Claim 2. For an odd cut set X of minimum cardinality minimizing $|E(G_X)|$, the following properties hold:

(B1) there does not exist three vertices $u, v, w \in X$ such that $N(u) \cap N(v) \cap N(w) \neq \emptyset$ and $\max(d_{G_X}(u), d_{G_X}(v), d_{G_X}(w)) \geq 3$;

(B2) if there is an induced K_4 in G_X, then at most two vertices of U are adjacent in G;

(B3) if there is an induced K_4 in G_X and two vertices of U are adjacent in G, then no vertex of U is in an induced triangle in G;

(B4) if there is an induced K_4 in G_X and two vertices of U are adjacent in G, then for every vertex $v \in \{u_{12}, u_{13}, u_{14}, u_{23}, u_{24}, u_{34}\}$, we have $|B_2(v) \cap X| \leq 4$;

(B5) if there is an induced K_4 in G_X and U is an independent set in G, then for every vertex $v \in \{u_{12}, u_{13}, u_{14}, u_{23}, u_{24}, u_{34}\}$, we have $|B_2(v) \cap X| \leq 3$;

(B6) if there is an induced K_4 in G_X and U is an independent set in G, then the vertices of at most one pair among $\{(u_{12}, u_{34}), (u_{13}, u_{24}), (u_{14}, u_{23})\}$ have a common neighbor.

Proof.

(B1): Let u, v and w be three distinct vertices of X and let z be a common neighbor of these three vertices. Suppose $d_{G_X}(u) \geq 3$. Notice that, for any odd cycle C_u, $z \notin C_u$, since it would imply that either v or w belongs to $V(C_u)$. Thus, there exist two distinct vertices u_1

and u_2 such that $\{u_1, u_2\} \subseteq (N(u) \setminus \{z\}) \cap V(C_u)$. Moreover, both u_1 and u_2 have at least two neighbors in $V(C_u)$. Since $d_{G_X}(u) \geq 3$, either u_1 or u_2 has a neighbor in X. Suppose, without loss of generality, that u_1 has a neighbor $u' \in X$. In this case, we can also suppose that there exists one odd cycle $C_{u'}$ such that $u_1 \notin V(C_{u'})$. If such odd cycle did not exist, then $(X \setminus \{u, u'\}) \cup \{u_1\}$ would be also an odd-cut set, contradicting the minimality of X. Let u'' be the neighbor of u_1 in $V(C_u) \setminus \{u\}$. Since u' has two neighbors in $V(C_{u'})$, u' has no neighbors in X and since u'' has two neighbors in $V(C_u)$, we obtain that $|(B_2(u_1) \cap X) \setminus \{u\}| \leq 2$. Consequently, $(X \setminus \{u\}) \cup \{u_1\}$ is also a odd cut set, and since $\{u_1v, u_1w\} \notin E(G_{(X\setminus\{u\})\cup\{u_1\}})$ and we added at most one edge in $E(G_{(X\setminus\{u\})\cup\{u_1\}})$, we have $|E(G_{(X\setminus\{u\})\cup\{u_1\}})| < |E(G_X)|$, contradicting the minimality of $|E(G_X)|$.

(B2): Notice that by Property (B1), no three vertices of U have a common neighbor. Moreover, by Property (A1), a vertex of X cannot have two neighbors in X. Consequently, at most two pairs of vertices of U are adjacent. Suppose, without loss of generality, that (u_1, u_2) and (u_3, u_4) are two pairs of adjacent vertices. In this case, we remove U from X and replace it by $\{u_{13}, u_{14}, u_{24}\}$, contradicting the minimality of X.

(B3): Suppose, without loss of generality, that u_{12} and u_{13} exist and are adjacent. Let u' and u'' be two existing vertices among $\{u_{14}, u_{23}, u_{24}, u_{34}\}$ such that the three vertices u', u'' and u_{13} have no common neighbor. In this case, we remove U from X and replace it by $\{u', u'', u_{13}\}$, contradicting the minimality of X. There exist three configurations depending on which are the two adjacent vertices from U. However, for the three possibilities, it is trivial to check that if we remove U from X and replace it by $\{u', u'', u_{13}\}$ we obtain an odd-cut set.

(B4): Suppose, without loss of generality, that u_1 and u_2 are adjacent. We can remark that, by Property (B1), no vertex of $\{u_{13}, u_{14}, u_{23}, u_{24}, u_{34}\}$ has a neighbor in $X \setminus U$. A consequence is that $|B_2(u_{34}) \cap X| \leq 4$. Suppose that $|B_2(u_{13}) \cap X| > 4$. Then, X contains two neighbors of u'_{13}. Consequently, every odd cycle C with $u_1 \in V(C)$ contains also another vertex of X. Hence, $X \setminus \{u_1\}$ is an odd-cut set, contradicting the minimality of X. By symmetry, the same goes for u_{14}, u_{23} and u_{24}.

(B5): Suppose that the vertex u_{12} satisfies $|B_2(u_{12}) \cap X| > 3$. We can remark that if u_{13} satisfied also $|B_2(u_{13}) \cap X| > 3$, then, as for Property (B4), $X \setminus \{u_1\}$ would be an odd-cut set, contradicting the minimality of X. Thus, we can suppose that $|B_2(u_{13}) \cap X| \leq 3$. Consequently, we can remove $\{u_1\}$ from X and replace it by $\{u_{13}\}$, contradicting the minimality of $|E(G_X)|$. By symmetry, the same property holds for u_{13}, u_{14}, u_{23}, u_{24} and u_{34}.

(B6): Suppose that the vertices of two pairs among $\{(u_{12}, u_{34}), (u_{13}, u_{24}), (u_{14}, u_{23})\}$ have a common neighbor and that these two pairs are (u_{12}, u_{34}) and (u_{13}, u_{24}), i.e. $N(u_{12}) \cap N(u_{34}) \neq \emptyset$ and $N(u_{13}) \cap N(u_{24}) \neq \emptyset$ (other pairs can be treated similarly). Notice that $(X \setminus U) \cup \{u'_{12}, u_{14}, u_{23}\}$ is also an odd-cut set, contradicting the minimality of X. Note that in this case, there is an even cycle containing each vertex of U that does not contain vertices of $(X \setminus U) \cup \{u'_{12}, u_{14}, u_{23}\}$.

□

Now suppose that X is an odd-cut set in G of minimum cardinality minimizing $|E(G_X)|$. We consider two types of induced K_4 in G_X. The first type is a K_4 such that U is an independent set in G and that both (u_{12}, u_{34}) and (u_{14}, u_{23}) are pairs of vertices with no common neighbor. The second type is a K_4 such that u_1 and u_2 are adjacent in G. By Properties (B2), (B3) and (B6), every induced K_4 in G_X is isomorphic to an induced K_4 of the first or second type. We construct Y by replacing U by $\{u_{12}, u_{23}, u_{34}, u_{41}\}$ in X, each time there is an induced K_4 of the first type in G_X and by replacing U by $\{u_2, u_3, u_4, u_{13}\}$ in

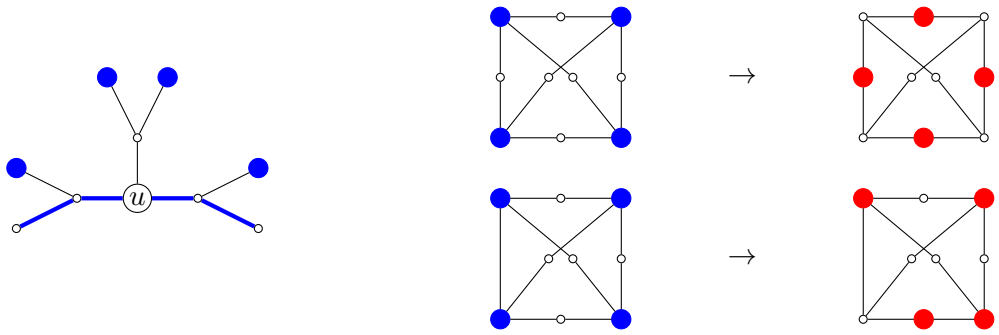


Figure 6: A vertex $u \in X$ satisfying $d_{G_X}(u) = 4$ (on the left) and four vertices of X forming an induced K_4 in G_X (on the middle) and their corresponding vertices in Y (on the right) (bold vertices: vertices from X (on the left and middle) or Y (on the right), thick edges: edges belonging to C_u).

X, each time there is an induced K_4 of the second type in G_X. The middle part of Figure 6 illustrates the two types of induced K_4 in G_X and the right part illustrates the corresponding vertices in Y. Notice that, by Properties (B4) and (B5), the graph G_Y has no more induced K_4.

Now, it remains to prove that Y is an odd-cut set and that G_Y is subcubic. Suppose there is an induced K_4 in G_X. If this K_4 is of the first type, then, since the set $\{u_{12}, u_{23}, u_{34}, u_{41}\}$ contains two neighbors of each vertex from U, we obtain that if u_i belongs to an odd cycle, then at least one vertex from $\{u_{12}, u_{23}, u_{34}, u_{41}\}$ belongs to this odd cycle. If this K_4 is of the second type, notice that if u_1 belongs to an odd cycle, then at least one vertex from $\{u_2, u_{13}\}$ belongs to this odd cycle. Consequently, Y is an odd-cut set and since X has the same cardinality than Y, Y has minimum cardinality and satisfies Property (A4). Moreover, by Properties (B4) and (B5), we obtain that no vertex of $Y \setminus X$ has a common neighbor with two other vertices of Y and consequently that Y satisfies Property (B1). Hence, by Property (A4), we obtain that G_Y is subcubic and has no connected component isomorphic to K_4, which concludes the proof.

□

Remark that in the previous proof, except in the proof of Property (B6), we did not use the fact that the cycles have odd length. Thus, using a similar proof, we can possibly obtain that for any subcubic graph G, there exists a minimum cut set that can be partitioned in three disjoint 2-packings of G.

The Petersen graph is an example of cubic graph which is not $(1, 1, k, k')$-colorable, for any $k, k' \geq 2$, showing that the result of Theorem 3 is tight in a certain sense.

Proposition 4. *For any $k, k' \geq 2$, the Petersen graph is not $(1, 1, k, k')$-colorable.*

Proof. Since the diameter of the Petersen graph is 2, for any $k \geq 2$, a color k can be used to color only one vertex. Hence, it suffices to prove that we can color at most seven vertices with the two colors 1. A maximum independent set in the Petersen graph contains at most four vertices and an independent set of four vertices is made of two vertices of the outer cycle and two vertices of the inner cycle of the Petersen graph. We can remark that for every

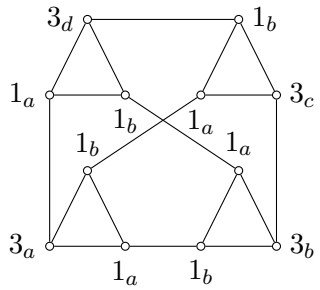


Figure 7: A cubic non $(1,1,3,3,3)$-colorable graph of order 12.

possible independent set of four vertices A, there are at most three non-adjacent vertices not belonging to A. □

The experiments suggest that the Petersen graph could be the only non $(1,1,2,3)$-colorable subcubic graph, see Table 2. Furthermore, the next result shows that the three colors 2 cannot be replaced by three colors 3 in Theorem 3.

Proposition 5. *There exist cubic graphs that are not* $(1,1,3,3,3)$*-colorable.*

Proof. Consider the cubic graph depicted in Figure 7. Since it has diameter 3, no more than one vertex could be colored by a color 3. Moreover, it contains four disjoint triangles and each triangle should contain one vertex not colored by 1. Thus, it is impossible to color it with the sequence $(1,1,3,3,3)$. □

$n\backslash S$	$(1,1)$	$(1,1,2)$	$(1,1,2,3)$	$(1,1,2,3,3)$
4	0	0	1	0
6	1	0	1	0
8	1	2	2	0
10	2	9	7	1
12	5	42	38	0
14	13	314	182	0
16	38	2808	1214	0
18	149	32766	8386	0
20	703	423338	86448	0
22	4132	6212201	1103114	0

Table 2: Number of S-chromatic cubic graphs of order n up to 22.

We now show that 3-irregular subcubic graphs are $(1,1,2)$-colorable. Notice that the subdivided graph $S(G)$ of any graph G is $(1,1)$-colorable as it is bipartite.

Theorem 4. *Every 3-irregular subcubic graph is* $(1,1,2)$*-colorable.*

Proof. Let G be a 3-irregular graph and let $e = xy$ be any edge of G such that x and y both have degree at most 2. If no such edge exist then the graph is bipartite and consequently $(1,1)$-colorable.

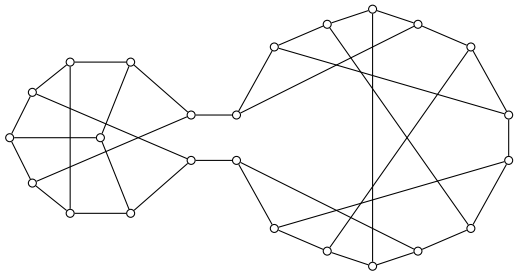


Figure 8: A cubic graph of order 24 with packing chromatic number 11.

Define a level ordering L_i, $0 \leq i \leq r = \epsilon(e)$, of (G, e).

We first construct a coloring c of the vertices of G from level r to 1 and with colors from the set $\{1_a, 1_b, 2\}$, that satisfies the following property:

i) No vertex of degree at most 2 is colored 2.

The set L_r induces a disjoint union of paths of order at most 3 in G. Since paths are $(1,1)$-colorable, L_r is $(1,1)$-colorable. Thus, Property i) is satisfied.

Assume that we have already colored all vertices of G of levels from r to $i+1$ and that we are going to color vertex $u \in L_i$, $1 \leq i \leq r-1$. If u has degree at most 2, then $\{1_a, 1_b\} \not\subseteq C_1(u)$. Hence, u can be colored by 1_a or 1_b and Property i) is satisfied. If u has degree 3 and if $C_1(u) \neq \{1_a, 1_b\}$, then u can be colored by 1_a or 1_b. Else if $C_1(u) = \{1_a, 1_b\}$, let u_1 and u_2 be the colored neighbors of u, with $c(u_1) = 1_a$ and $c(u_2) = 1_b$. The vertex u_1 has a neighbor colored by 1_b and the vertex u_2 has a neighbor colored by 1_a, if not u_1 and u_2 could be recolored and u could be colored by 1_a or 1_b. Thus, we have $C_1(u) \cup C_2(u) = \{1_a, 1_b\}$ because G is 3-irregular and we can color u by the color 2.

Finally, it remains to color vertices of L_0, i.e. x and y. If $1_a \notin C_1(x)$ and $1_b \notin C_1(y)$ (or, symmetrically, $1_b \notin C_1(x)$ and $1_a \notin C_1(y)$). Then set $c(x) = 1_a$ and $c(y) = 1_b$ (or, symmetrically, $c(x) = 1_b$ and $c(y) = 1_a$). Let x_1 be the possible neighbor of x different from y and let y_1 be the possible neighbor of y different from x. Without loss of generality, suppose that $c(x_1) = 1_a$ and $c(y_1) = 1_a$. Suppose that x_1 has degree at most 2. If $2 \in C_1(x_1)$, then x_1 can be recolored by 1_b and we can set $c(x) = 1_a$ and $c(y) = 1_b$. Else, $C_1(x_1) = \{1_b\}$ and we can set $c(x) = 2$ and $c(y) = 1_b$. If x_1 has degree 3, then every colored neighbor of x has at most degree 2 and is colored by 1_b by Property i). Thus, since $2 \notin C_1(x_1)$, we can set $c(x) = 2$ and $c(y) = 1_b$. Therefore, we obtain a $(1,1,2)$-coloring of G. □

4 $(1, 2, 3, \ldots)$-coloring

The question of whether cubic graphs have finite packing chromatic number or not was raised by Goddard et al. [12]. We give some partial results related to this question.

For the subdivision of a cubic graph, Proposition 1 implies that if every subcubic graph G different from the Petersen graph is $(1,1,2,2)$-colorable, then $S(G)$ is $(1,3,3,5,5)$-colorable and consequently $\chi_\rho(S(G)) \leq 5$. On the other side, it can be easily verified that $\chi_\rho(S(K_4)) = 5$.

For arbitrary cubic graphs, we can (only) state the following:

Proposition 6. *There exists a cubic graph with packing chromatic number* 13.

Proof. The cubic graph of order 38 and diameter 4 (which is a largest cubic graph with diameter 4) described independently in [1, 19] needs 13 colors to be packing colored (checked by computer). By running a brute force search algorithm, we found that at most 28 vertices can be colored with colors $\{1, 2, 3\}$. But, since this graph has diameter 4, then every color greater than 3 can be given to only one vertex, implying the use of all colors from $\{4, \ldots, 13\}$ to complete the coloring. □

The distribution of packing chromatic numbers for cubic graphs of order up to 20 is presented in Table 3. With the help of a computer, we also found a cubic graph of order 24 and packing chromatic number 11. This graph is illustrated in Figure 8.

$n \setminus \chi_\rho$	4	5	6	7	8	9	10	11
4	1	0	0	0	0	0	0	0
6	1	1	0	0	0	0	0	0
8	0	3	2	0	0	0	0	0
10	0	3	15	1	0	0	0	0
12	0	7	42	36	0	0	0	0
14	0	13	252	222	22	0	0	0
16	0	34	907	2685	433	1	0	0
18	0	116	5277	21544	14050	314	0	0
20	0	151	22098	206334	226622	55284*		0

Table 3: Number of cubic graphs of order n with packing chromatic number χ_ρ up to 20.*There are 55284 cubic graphs of order 20 and with packing chromatic number between 9 and 10 (our program takes too long time to compute their packing chromatic numbers).

5 Concluding remarks

We conclude this paper by listing a few open problems:

- Is it true that any subcubic graph except the Petersen graph is $(1, 1, 2, 3)$-colorable?
- Is it true that any subcubic graph except the Petersen graph is $(1, 2, 2, 2, 2, 2)$-colorable?
- Does there exist a 3-irregular subcubic graph that is not $(1, 2, 2, 3)$-colorable?
- Is it true that any 3-irregular subcubic graph is $(1, 1, 3)$-colorable?
- Is it true that the subdivision of any subcubic graph is $(1, 2, 3, 4, 5)$-colorable?
- Does there exist a cubic graph with packing chromatic number larger than 13?

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