S-Packing Colorings of Cubic Graphs

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Given a non-decreasing sequence $S = (s_1, s_2, \ldots, s_k)$ of positive integers, an *S-packing coloring* of a graph G is a mapping c from $V(G)$ to $\{s_1, s_2, \ldots, s_k\}$ such that any two vertices with color s_i are at mutual distance greater than s_i, $1 \leq i \leq k$. This paper studies S-packing colorings of (sub)cubic graphs. We prove that subcubic graphs are $(1, 2, 2, 2, 2, 2, 2)$-packing colorable and $(1, 1, 2, 2, 3)$-packing colorable. For subdivisions of subcubic graphs we derive sharper bounds, and we provide an example of a cubic graph of order 38 which is not $(1, 2, \ldots, 12)$-packing colorable.

Keywords: coloring, packing chromatic number, cubic graph.

1 Introduction

A proper coloring of a graph G is a mapping which associates a color (integer) to each vertex such that adjacent vertices get distinct colors. In such a coloring, the color classes are stable sets (1-packings). As an extension, a d-distance coloring of G is a proper coloring of the d-th power G^d of G, i.e. a partition of $V(G)$ into d-packings (sets of vertices at pairwise distance greater than d). While Brook's theorem implies that all cubic graphs except the complete graph K_4 of order 4 are properly 3-colorable, many authors studied 2-distance colorings of cubic graphs.

The aim of this paper is to study a mixing of these two types of colorings, i.e. colorings of (sub)cubic graphs in which some colors classes are 1-packings while other are d-packings, $d \geq 2$. Such colorings can be expressed using the notion of S-packing coloring: For a non-decreasing sequence $S = (s_1, s_2, \ldots, s_k)$ of positive integers, an *S-packing coloring* (or simply S-coloring) of a graph G is a coloring of its vertices with colors from

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$\{s_1, s_2, \ldots, s_k\}$ such that any two vertices with color s_i are at mutual distance greater than s_i, $1 \leq i \leq k$. The color class of each color s_i is thus an s_i-packing. The graph G is *S-colorable* if there exists an S-coloring and it is *S-chromatic* if it is S-colorable but not S'-colorable for any $S' = (s_1, s_2, \ldots, s_j)$ with $j < k$ (note that Goddard et al. [13] define differently the S-chromaticness for infinite graphs).

A $(d, \ldots, d)$-coloring is thus a d-distance k-coloring, where k is the number of d (see [16] for a survey of results on this invariant) while a $(1, 2, \ldots, d)$-coloring is a packing coloring. The packing chromatic number $\chi_\rho(G)$ of G is the integer k for which G is $(1, \ldots, k)$-chromatic. This parameter was introduced recently by Goddard et al. [11] under the name of *broadcast chromatic number* and the authors showed that deciding whether $\chi_\rho(G) \leq 4$ is NP-hard. A series of works [3, 5, 7, 8, 11, 17] considered the packing chromatic number of infinite grids. For sequences S other than $(1, 2, ..., k)$, S-packing colorings were considered more recently [6, 10, 12, 13].

Regarding cubic graphs, the packing chromatic number of the hexagonal lattice and of the infinite 3-regular tree is 7 and at most 7, respectively. Goddard et al. [11] asked what is the maximum of the packing chromatic number of a cubic graph of order n. For 2-distance coloring of cubic graphs, Cranston and Kim have recently shown [4] that any subcubic graph is $(2, 2, 2, 2, 2, 2, 2, 2)$-colorable (they in fact proved a stronger statement for list coloring). For planar subcubic graphs G, there are also sharper results depending on the girth of G [2, 4, 15].

In this paper, we study S-packing colorings of subcubic graphs for various sequences S starting with one or two '1'. We also compute the distribution of S-chromatic cubic graphs up to 20 vertices, for three sequences S. The corresponding results are reported on Tables 1, 2, and 3. They are obtained by an exhaustive search, using the lists of cubic graphs maintained by Gordon Royle [14]. The paper is organized as follows: Section 2 is devoted to the study of $(1, k, \ldots, k)$-colorings of subcubic graphs for $k = 2$ or 3; Section 3 to $(1, 1, 2, \ldots)$-colorings; Section 4 to $(1, 2, 3, \ldots)$-colorings and Section 5 concludes the paper by listing some open problems.

1.1 Notation

To describe an S-coloring, if an integer s is repeated in the sequence S, then we will denote the colors s by $s_a, s_b, \ldots$.

The *subdivided graph* $S(G)$ of a (multi)graph G is the graph obtained from G by subdividing each edge once, i.e. replacing each edge by a path of length two. In $S(G)$, vertices of G are called *original* vertices and other vertices are called *subdivision* vertices. Let us call a graph *d-irregular* if it has no adjacent vertices of degree d. Note that graphs obtained from subcubic graphs by subdividing each edge at least once are 3-irregular graphs.

The following method (that is inspired from that of Cranston and Kim [4]) is used in the remainder of the paper to produce a desired coloring of a subcubic graph: for a graph G and an edge $e = xy \in E(G)$, a *level ordering* of (G, e) is a partition of $V(G)$ into levels $L_i = \{v \in V(G) : d(v, e) = i\}$, $0 \leq i \leq \epsilon(e)$, with $\epsilon(e) = \max(\{d(u, e), u \in V(G)\}) \leq diam(G)$. The vertices are then colored one by one, from level $\epsilon(e)$ to 1, while

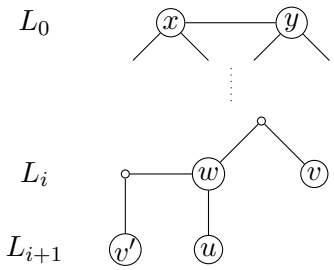

Figure 1: Level ordering of a subcubic graph from the edge xy. The vertices v, w are siblings and v and v' are both cousins of u.

preserving some properties. These properties are used at the end to allow to color the vertices x and y by recoloring possibly some vertices in their neighborhood.

Two vertices u and v of G are called *siblings* if they are not adjacent, are on the same level L_i for some $i \geq 1$ and have a common neighbor in L_{i-1}. Two vertices u and v of G are called *cousins* if they are at distance 3 and in every path of length 3 between u and v, there is a neighbor of u or v in a lower level than the level of u and v. Note that a vertex has at most one sibling and two cousins (see Figure 1). Given a (partial) coloring c of G, let $C_1(u) = \{c(v) : uv \in E(G)\}$, $C_2(u) = \{c(v) : d(u,v) = 2, \text{ with } u, v \text{ not siblings}\}$, $C_3(u) = \{c(v) : d(u,v) = 3, \text{ with } u, v \text{ not cousins}\}$, and $\tilde{C_3}(u) = \{c(v) \in \{2_a, 2_b, 3\}, \text{ with } u, v \text{ cousins}\}$.

2 $(1, k, \ldots, k)$-coloring

In this section, $(1, k, \ldots, k)$-colorings of subcubic graphs are studied for $k = 2$ or 3.

2.1 $(1, 3, \ldots, 3)$-coloring

The following proposition is used to obtain an S-coloring of a subdivided graph:

Proposition 1. *Let G be a graph and $S = (s_1, \ldots, s_k)$ be a non-decreasing sequence of integers. If G is S-colorable then $S(G)$ is $(1, 2s_1 + 1, \ldots, 2s_k + 1)$-colorable.*

Proof. Let c be an S-coloring of G. Every pair of vertices $u, v \in V(G)$ such that $d(u,v) = d$ become at distance $2d + 1$ in $S(G)$. Therefore, every set of vertices in $V(G)$ forming an i-packing also forms a $2i + 1$-packing in $S(G)$. Using color 1 on subdivision vertices and using the coloring c (considering the sequence differently) on original vertices, we obtain a $(1, 2s_1 + 1, \ldots, 2s_k + 1)$-coloring of $S(G)$. □

Corollary 1. *For every subcubic graph G, $S(G)$ is $(1, 3, 3, 3)$-colorable.*

Proof. Brooks' theorem asserts that every subcubic graph except K_4 is $(1, 1, 1)$-colorable. Hence, by Proposition 1, every subcubic graph G except K_4 is such that $S(G)$ is

$(1,3,3,3)$-colorable. We define a $(1,3,3,3)$-coloring of $S(K_4)$ as follows: let $\gamma : E(K_4) \to \{a,b,c\}$ be a proper edge 3-coloring of K_4. Put color 1 on all four original vertices of K_4 and put color $3_{\gamma(e)}$ on each subdivision vertex corresponding with edge e of K_4. □

Goddard et al. [11] characterized $(1,3,3)$-colorable graphs as the graphs obtained from any bipartite multigraph by subdividing it and adding leaves on original vertices. Therefore, there are many subdivided subcubic graphs that are not $(1,3,3)$-colorable (for instance $S(C_3) = C_6$), showing that the bound of Corollary 1 is tight in a certain sense.

2.2 $(1,2,\ldots,2)$-coloring

Note that no cubic graph with more than 3 vertices is $(1,2,2)$-colorable since a graph with three vertices of degree larger than 2 at mutual distance less than 2, is not $(1,2,2)$-colorable. However, there exist $(1,2,2)$-colorable subcubic graphs and it has been recently proved [9] that determining if a subcubic bipartite graph is $(1,2,2)$-colorable is NP-complete.

Proposition 2. *Every subcubic graph is* $(1,2,2,2,2,2,2)$*-colorable.*

Proof. Let G be a subcubic graph and let $e = xy$ be any edge of G. Define a level ordering L_i, $0 \leq i \leq r = \epsilon(e)$, of (G,e).

We first construct a coloring c of the vertices of G from level r to 1 and with colors from the set $C = \{1, 2_a, 2_b, 2_c, 2_d, 2_e, 2_f\}$, that satisfies the following properties :

i) color 1 is used as often as possible, i.e. when coloring a vertex u, if no neighbor is colored 1, then u is colored 1;

ii) if u is colored 2, then there is subsidiary color $\tilde{c}(u) \in C$ different from $c(u)$ such that $\tilde{c}(u) \notin C_1(u) \cup C_2(u)$, but with possibly $\tilde{c}(u) = c(v)$ if u and v are siblings.

The set L_r induces a disjoint union of paths and cycles in G. Since paths and cycles are $(1,2,2,2)$-colorable, we are able to construct a coloring of the vertices of L_r as follows. Start by coloring each path/cycle with colors $\{1, 2_a, 2_b, 2_c\}$. For each pair of vertices u, v in different paths/cycles at distance 2 both colored 2_a (2_b or 2_c respectively), set $c(u) = 2_d$ (2_e or 2_f, respectively). Afterwards, for every vertex u of colors 2_a (2_b, 2_c, 2_d, 2_e or 2_f, respectively), set $\tilde{c}(u) = 2_d$ (2_e, 2_f 2_a, 2_b or 2_c, respectively). Then Property ii) is satisfied.

Assume that we have already colored all vertices of G of levels from r to $i+1$ and that we are going to color vertex $u \in L_i$, $1 \leq i \leq r-1$. If $1 \notin C_1(u)$ then set $c(u) = 1$ (Property i) is then satisfied). Now, if $1 \in C_1(u)$, then let u_1 be the neighbor of u of color 1 and let u_2 be the other neighbor of u, if any. By construction, either $c(u_2) = 1$ or $1 \in C_1(u_2)$, hence $|C_1(u) \cup C_2(u)| \leq 5$. In that case there are at least two colors $\{2_\alpha, 2_\beta\} \subset C \setminus C_2(u)$ for some $\alpha, \beta \in \{a, \ldots, f\}$, with possibly, if u has a sibling v, $2_\beta = c(v)$. Then set $c(u) = 2_\alpha$ and $\tilde{c}(u) = 2_\beta$ (Property ii) is then satisfied). Figure 2 illustrates this case.

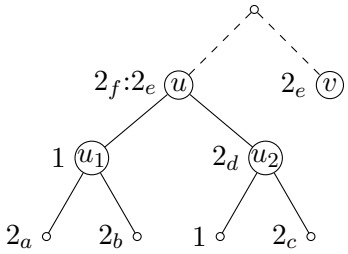

Figure 2: A configuration in the proof of Proposition 2, when coloring vertex u. The label 2_f:2_e on u means that $c(u) = 2_f$ and $\tilde{c}(u) = 2_e$.

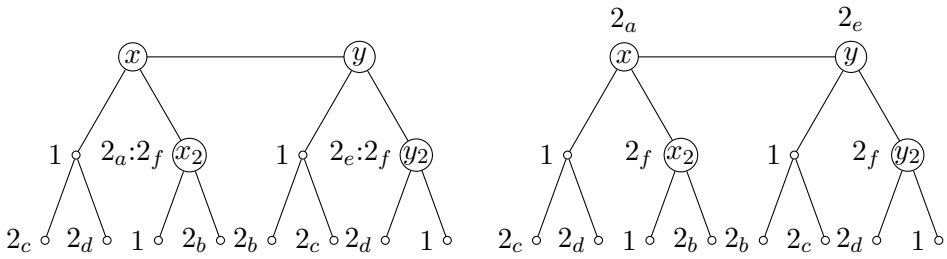

Figure 3: A configuration in the proof of Proposition 2, before (on the left) and after (on the right) coloring x and y.

Finally, it remains to color vertices of L_0, i.e., x and y. If $1 \in C_1(x) \cap C_1(y)$ then, by Property i), the neighbor x_2 of x colored 2, if any, has a neighbor of color 1 and the same goes for y, with y_2 being the neighbor of y colored 2, if any. Hence $|C_1(x) \cup C_2(x)| \leq 5$, $|C_1(y) \cup C_2(y)| \leq 5$ and there remains at least a color 2_α available for x and a color 2_β for y. If $\alpha = \beta$ then set $c(x) = c(x_2)$, $c(y) = c(y_2)$, $c(x_2) = \tilde{c}(x_2)$ and $c(y) = \tilde{c}(y_2)$. Figure 3 illustrates that case.

If $1 \in C_1(x)$ but $1 \notin C_1(y)$ (or $1 \in C_1(y)$ but $1 \notin C_1(x)$, by symmetry), then set $c(y) = 1$ and if $C_2(x) = C$ then set $c(x) = c(x_2)$ and $c(x_2) = \tilde{c}(x_2)$, else give to x an available color.

Otherwise, $1 \notin C_1(x) \cup C_1(y)$. Then set $c(y) = 1$ and we show that there is always a color 2 to assign to x. If $|C_1(x) \cup C_2(x)| \leq 6$ then there is a color available for x. Else, let x_1, x_2 be the two neighbors of x other than y and let x'_1 (x'_2, respectively) be the neighbor of x_1 (x_2, respectively) colored 2 other than x (no more than one, as x_1 and x_2 both have a neighbor colored 1). Suppose, without loss of generality, that $c(x_1) = 2_a$, $c(x_2) = 2_b$, $c(x'_1) = 2_c$ and $c(x'_2) = 2_d$. If $\tilde{c}(x_1) \in \{2_d, 2_e, 2_f\}$ then recolor x_1 by its subsidiary color $\tilde{c}(x_1)$ and set $c(x) = 2_a$. Similarly, if $\tilde{c}(x_2) \in \{2_c, 2_e, 2_f\}$ then recolor x_2 by its subsidiary color $\tilde{c}(x_2)$ and set $c(x) = 2_b$. Else, $\tilde{c}(x_1) = 2_b$ and $\tilde{c}(x_2) = 2_a$. Recolor x'_1 by its subsidiary color $\tilde{c}(x'_1)$ and set $c(x) = 2_c$. If $\tilde{c}(x'_1) = 2_a = c(x_1)$, then switch the colors of x_1 and x_2 (this is possible since $\tilde{c}(x_1) = c(x_2)$ and $\tilde{c}(x_2) = c(x_1)$). Figure 4

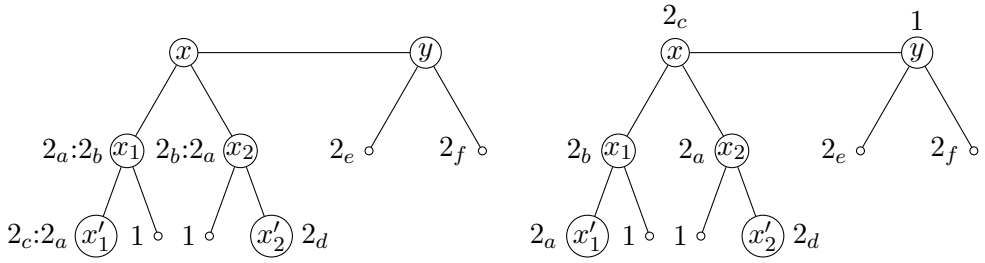

Figure 4: A configuration in the proof of Proposition 2, before (on the left) and after (on the right) coloring x and y.

illustrates this case.

Therefore, we obtain, in all cases, a $(1, 2, 2, 2, 2, 2, 2)$-coloring of G. □

The Petersen graph is an example of cubic graph which is not $(1, 2, 2, 2, 2, 2)$-colorable, showing that the result of Proposition 2 is tight in a certain sense. However, the experiments reported on Table 1 suggest that the Petersen graph could be the only non $(1, 2, 2, 2, 2, 2)$-colorable subcubic graph.

$n \backslash S$	$(1, 2, 2, 2)$	$(1, 2, 2, 2, 2)$	$(1, 2, 2, 2, 2, 2)$	$(1, 2, 2, 2, 2, 2, 2)$
4	1	0	0	0
6	1	1	0	0
8	2	1	2	0
10	11	7	0	1
12	11	74	0	0
14	254	250	5	0
16	1031	3017	12	0
18	15960	25297	44	0
20	178193	332045	251	0
22	2481669	4835964	1814	0

Table 1: Number of S-chromatic cubic graphs of order n up to 22.

Furthermore, as the following proposition shows, even some bipartite cubic graphs are not $(1, 2, 2, 2, 2, 3)$-colorable.

Proposition 3. *There exist bipartite cubic graphs that are not* $(1, 2, 2, 2, 2, 3)$*-colorable.*

Proof. The cubic graph depicted in Figure 5 is bipartite and is $(1, 2, 2, 2, 2, 2)$-colorable, as shown on the figure. Let (A, B) be the two subsets of vertices that form a bipartition of this graph. Suppose this graph is $(1, 2, 2, 2, 2, 3)$-colorable and let c be a $(1, 2, 2, 2, 2, 3)$-coloring and X_1 be the set of vertices colored 1. Remark that the cardinality of any 2-packing is at most 2 and that any pair of vertices (u, v) included in A or in B is such

that $d(u, v) \leq 2$. We have $|X_1| \geq 5$, as at most one vertex can be colored 3 (since the diameter of the graph is 3) and at most two vertices can be colored the same color 2.

Firstly, if $X_1 \subseteq A$ or $X_1 \subseteq B$, then each remaining vertex should be colored differently in the other partition, which is impossible since $|A| = |B| = 7$.

Secondly, if there are vertices colored 1 in A and B, then the only possibility in order to have $|X_1| \geq 5$ is to have one vertex colored 1 in one partition and four vertices colored 1 in the other partition. Suppose, without loss of generality, that $|X_1 \cap A| = 1$ and $|X_1 \cap B| = 4$. Exactly three vertices are not colored 1 in B. Consequently, only three pairs of vertices can have the same color 2 and the nine vertices not colored 1 cannot be all colored with the remaining colors 2 and 3. □

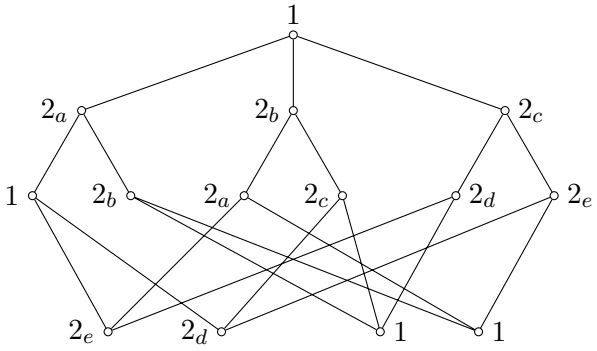

Figure 5: A cubic bipartite $(1, 2, 2, 2, 2, 2)$-chromatic graph of order 14.

The next results show that there are sub-families of subcubic graphs that can be colored with fewer colors.

Proposition 4. *Every 3-irregular subcubic graph is* $(1, 2, 2, 2)$*-colorable.*

Proof. Let G be a 3-irregular graph and let $e = xy$ be any edge of G such that x and y are both of degree at most 2. If no such edge exists, then the graph is the subdivision $S(H)$ of some subcubic graph H where leaves could be added on original vertices of degree 2 and thus G is $(1, 3, 3, 3)$-colorable by Corollary 1. Define a level ordering L_i, $0 \leq i \leq r = \epsilon(e)$, of (G, e).

We construct a coloring c of the vertices of G from level r to 1 and with colors from the set $\{1, 2_a, 2_b, 2_c\}$, that satisfies the following properties :

i) color 1 is used as often as possible, i.e. when coloring a vertex u, if no neighbor is colored 1, then u is colored 1;

ii) every vertex of degree 2 is colored 1 when first coloring vertices of L_i, except if the connected component containing this vertex in L_i is a path of order 2 (in which case one of the two vertices is colored 1).

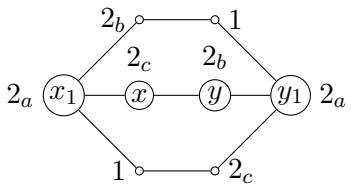

Figure 6: The graph $G_{(1,2,2,2)}$ from Proposition 4.

The set L_r induces a disjoint union of paths of order at most 3 in G. Since paths are $(1,2,2)$-colorable, L_r is $(1,2,2)$-colorable. Moreover, in every path of order 3 in L_r, the central vertex has degree 3, thus a color 1 could be given to every vertex of degree 2. If the path is of order 2, one of the vertex is colored 1. Thus, Properties i) and ii) are satisfied.

Assume that we have already colored all vertices of G of levels from r to $i+1$ and that we are going to color vertex $u \in L_i$, $1 \leq i \leq r-1$. We consider two cases depending on the degree of u:

Case 1. u is of degree 3.

If $1 \notin C_1(u)$, then u can be colored 1. Let u_1 and u_2 be the colored neighbors of u, with $c(u_1) = 1$. By Property i), a colored neighbor of u_2 has color 1. Hence, we have $|C_1(u) \cup C_2(u)| \leq 3$ and u can be colored some color 2.

Case 2. u is of degree at most 2.

Let u_1 be the colored neighbor of u, if any. If u_1 is of degree 3, let $u_{1,1}$ and $u_{1,2}$ be the colored neighbor of u, $u_{1,1,1}$ be the neighbor of $u_{1,1}$ different from u and $u_{1,2,1}$ be the neighbor of $u_{1,2}$ different from u. If $1 \notin C_1(u)$, then we can set $c(u) = 1$. Otherwise, $c(u_1) = 1$ and thus $c(u_{1,1}) \neq 1$ and $c(u_{1,2}) \neq 1$. Therefore, $c(u_{1,1,1}) = c(u_{1,2,1}) = 1$. Thus, u_1 can be recolored some color 2 and we can set $c(u) = 1$. If u_1 is of degree at most 2, then, as $|C_1(u_1) \cup C_2(u_1)| \leq 3$, we can recolor u_1 by a color 2. Thus, we can set $c(u) = 1$.

Finally, it remains to color vertices of L_0, i.e. x and y. Let x_1 be the possible neighbor of x different from y and let y_1 be the possible neighbor of y different from x. We consider three cases that cover all the possibilities by symmetry:

Case 1. x_1 and y_1 both have degree 3.

If their neighbors different from x and y are not adjacent between them, then, by Property ii), these vertices have color 1 and x_1 and y_1 have some color 2. Thus we can set $c(x) = 1$ and some color 2 to y, as $|C_1(y) \cup C_2(y)| \leq 3$. Suppose two pairs of neighbors of x_1 and y_1 different from x and y are adjacent. Thus, this graph is the graph $G_{(1,2,2,2)}$ from Figure 6 and is $(1,2,2,2)$-colorable. Suppose only two neighbors of x_1 and y_1 different from x and y are adjacent. Let $x_{1,1}$ and $y_{1,1}$ be these two neighbors, the other neighbors are colored 1 by Property ii). One of

these two vertices is colored 1 and the other one is colored 2. Suppose without loss of generality that $c(x_{1,1}) = 1$. Hence, we have $|C_1(x) \cup C_2(x)| \leq 3$ and we can color x by a color 2 and set $c(y) = 1$.

Case 2. x_1 has degree at most 2 and y_1 has degree 3.

By Property ii), x_1 is colored 1. As $|C_1(y_1) \cup C_2(y_1)| \leq 3$, then y_1 can be recolored some color 2. Thus, we can set $c(y) = 1$, and as $|C_1(x) \cup C_2(x)| \leq 3$ we can set a color 2 to x.

Case 3. x_1 and y_1 are both of degree at most 2.

If x_1 and y_1 are adjacent, then the graph is C_4 which is trivially $(1, 2, 2, 2)$-colorable. If x_1 and y_1 are not adjacent, then they both have color 1, $|C_1(x) \cup C_2(x)| \leq 2$ and $|C_1(y) \cup C_2(y)| \leq 2$. Thus, we can set some colors 2 to x and y.

Therefore, we obtain in all cases a $(1, 2, 2, 2)$-coloring of G. □

Remark that the 5-cycle C_5 is 3-irregular and is not $(1, 2, 2)$-colorable, hence the result of Proposition 4 is tight in a certain sense. However, there are 3-irregular subcubic graphs that are $(1, 2, 2, 3)$-colorable. The graph from Figure 6 is such an example (the vertex x can be recolored 1 and then color 2_c can be replaced by color 3).

We end this section with some results on subdivided graphs. Let $\delta(G)$ be the minimum degree of G.

Proposition 5. *For every graph G with $\delta(G) \geq 3$, if $S(G)$ is $(1, 2, 2)$-colorable then G is bipartite.*

Proof. Suppose $S(G)$ is $(1, 2, 2)$-colorable and G contains an odd cycle. In every $(1, 2, 2)$-coloring of a graph, every vertex of degree at least 3 should be colored some color 2 (if a vertex of degree at least 3 is colored 1, the coloring cannot be extended to the neighbors of this vertex). Therefore, if G contains an odd cycle, then $S(G)$ contains a cycle with an odd number of vertices of degree 3 and the colors 2_a and 2_b are not sufficient to alternately color these vertices. Hence $S(G)$ is not $(1, 2, 2)$-colorable. □

As every bipartite graph G is $(1, 1)$-colorable, then by Proposition 1, $S(G)$ is $(1, 3, 3)$-colorable (and also $(1, 2, 2)$-colorable and $(1, 2, 3)$-colorable). Thus, we obtain the following corollary.

Corollary 2. *For every graph G with $\delta(G) \geq 3$,*

$S(G)$ $(1, 2, 2)$-colorable $\Leftrightarrow$ $S(G)$ $(1, 2, 3)$-colorable $\Leftrightarrow$ $S(G)$ $(1, 3, 3)$-colorable $\Leftrightarrow$ G bipartite.

3 $(1, 1, 2, \ldots)$-coloring

Remind that bipartite graphs are $(1, 1)$-colorable. For non-bipartite subcubic graphs, we prove the following:

Proposition 6. *Every subcubic graph is* $(1,1,2,2,3)$*-colorable.*

Proof. Let G be a subcubic graph and let $e = xy$ be any edge of G. Define a level ordering L_i, $0 \leq i \leq r = \epsilon(e)$, of (G, e).

We first construct a coloring c of the vertices of G from level r to 1 and with colors from the set $\{1_a, 1_b, 2_a, 2_b, 3\}$, that satisfies the following properties :

i) colors 1 are used as often as possible, i.e. when coloring a vertex u, if no neighbor is colored 1_a then u is colored 1_a else if no neighbor is colored 1_b, then u is colored 1_b;

ii) if u is colored 2 or 3, then, except in the case where u and a sibling of u are both colored some color 2, u has a subsidiary color $\tilde{c}(u) \in \{2_a, 2_b, 3\}$ different from $c(u)$ such that:
   - $\tilde{c}(u) \notin C_1(u) \cup C_2(u) \cup C_3(u)$, but with possibly $\tilde{c}(u) \in \tilde{C}_3(v)$, if $\tilde{c}(u) = 3$;
   - $\tilde{c}(u) \notin C_1(u) \cup C_2(u)$, otherwise;

iii) if u is colored 3 then its sibling (if any) is colored 1_a or 1_b.

We begin with recalling that by Property i), vertices of color among $\{2_a, 2_b, 3\}$ have neighbors at the same level or at the above level colored 1_a and 1_b. Also, by properties ii) and iii), we have the following claim.

Claim 1. *If two cousins u and v are such that $\tilde{c}(u) = 3$ and $c(v) = 3$ and there is no vertex at distance at most 2 from v colored by $\{\tilde{c}(v)\} \cap \{2_a, 2_b\}$ in the below levels and no vertex at distance at most 3 from u colored by 3 in the below levels, then u and v can be recolored in order that $c(u) = 3$.*

The set L_r induces a disjoint union of paths and cycles in G. Since paths and cycles are $(1,1,2)$-colorable, we are able to construct a coloring of the vertices of L_r as follows. Start by coloring each path/cycle with colors $\{1_a, 1_b, 2_a\}$, using one color 2_a per odd cycle (and no color 2 for even cycle). For each pair of vertices in different paths/cycles at distance 2 both colored 2_a, recolor one of the vertex with color 2_b. Then, Properties i), ii) and iii) are satisfied.

Assume that we have already colored all vertices of G of levels from r to $i+1$ and that we are going to color vertex $u \in L_i$, $1 \leq i \leq r-1$. If $C_1(u) \neq \{1_a, 1_b\}$, then give to u the available color 1. Hence Property i) is satisfied and we can suppose that u has two neighbors in $L_i \cup L_{i+1}$, say u_1 and u_2, such that $c(u_1) = 1_a$ and $c(u_2) = 1_b$. Moreover, we can suppose that u_1 has a neighbor $u_{1,1}$ of color 1_b and u_2 has a neighbor $u_{2,1}$ of color 1_a since otherwise, u could be colored by either 1_a or 1_b after recoloring u_1 or u_2. Let $u_{1,2}$ be the other neighbor of u_1 different from u, if any, with $c(u_{1,2}) = \alpha$ and let $u_{2,2}$ be the other neighbor of u_2 different from u, if any, with $c(u_{2,2}) = \beta$. We consider three cases depending on the values of α and β:

Case 1. $\alpha = 1_b$ and $\beta = 1_a$. Then set $c(u) = 2_a$ or $c(u) = 2_b$ if the sibling of u is colored 2_a, and $\tilde{c}(u) = 2_b$ if u has no sibling colored 2. Thus Property ii) is satisfied.

Case 2. $\alpha = 1_b$ and $\beta \neq 1_a$ (or $\alpha \neq 1_b$ and $\beta = 1_a$, by symmetry). As $u_{2,2}$ has no sibling colored some color 2, then it has a subsidiary color by Property ii).

- If $c(u_{2,2}) \cup \tilde{c}(u_{2,2}) \neq \{2_a, 2_b\}$, then set $c(u_{2,2}) = 3$ (possibly, if $u_{2,2}$ has a cousin z colored 3, it can be seen that $u_{2,2}$ and z satisfy the conditions of Claim 1, hence z can be recolored by its subsidiary color). Thus we can set $c(u) = 2_a$ and $\tilde{c}(u) = 2_b$, or $c(u) = 2_b$ if the sibling of u is colored 2_a, and Property ii) is satisfied.
- If $c(u_{2,2}) \cup \tilde{c}(u_{2,2}) = \{2_a, 2_b\}$ and the sibling of u is colored 2, then give an appropriate color to $u_{2,2}$ in order u to have a color different from its sibling.
- If $c(u_{2,2}) \cup \tilde{c}(u_{2,2}) = \{2_a, 2_b\}$ and u has no sibling colored 2, suppose, without loss of generality, that $c(u_{2,2}) = 2_a$. As $u_{2,1}$ is colored 1_a, it has a neighbor colored 1_b different from u_2 (if not, $u_{2,1}$ and u_2 can be recolored and we can give a color 1 to u). If the other neighbor of $u_{2,1}$ is colored 3 then change its color by its subsidiary color using Claim 1. Since the vertices $u_{1,1}$ and $u_{1,2}$ are both colored 1_b, they could have two neighbors colored 2_a and 2_b or these vertices could have one neighbor colored 1_a different from u_1 and another neighbor different from u_1 of any color. In the latter case, recolor the possible neighbor of these vertices colored 3 by its subsidiary color using Claim 1. Then, set $c(u) = 2_b$ and $\tilde{c}(u) = 3$.

Case 3. $\alpha, \beta \in \{2_a, 2_b, 3\}$. Then the set $A = (c(u_{1,2}) \cup \tilde{c}(u_{1,2})) \cap (c(u_{2,2}) \cup \tilde{c}(u_{2,2}))$ is not empty. If $3 \in A$, then set $c(u_{1,2}) = 3$ and $c(u_{2,2}) = 3$ and set $c(u) = 2_a$ and $\tilde{c}(u) = 2_b$, or $c(u) = 2_b$, if the sibling of u is colored 2_a. Thus Property ii) is satisfied. Else, if $3 \notin A$ and u has a sibling colored 2, then change the color of $u_{1,2}$ and $u_{2,2}$ by appropriate colors and give a color 2 to u. Else, if $3 \notin A$ and u has no sibling colored 2, then give a color $2_\delta \in A$, with $\delta \in \{a, b\}$ to $u_{1,2}$ and $u_{2,2}$ and recolor each vertex at distance at most 3 from u by its subsidiary color using Claim 1 (there are at most two vertices colored 3 by hypothesis). Hence, set $c(u) = 2_\beta$ and $\tilde{c}(u) = 3$, $\beta \neq \alpha$ and $\beta \in \{a, b\}$, and Property ii) is satisfied. Property iii) is satisfied, as the color 3 has been given to vertices which have no sibling colored 2.

Now, it remains to color vertices of L_0, i.e., x and y. Let x_1 and x_2 be the possible neighbors of x different from y and y_1 and y_2 be the possible neighbors of y different from x. We consider seven cases that cover all the possible configurations for the colors of the neighbors of x and y (in order to simplify, configurations that can be obtained by exchanging x and y are omitted):

Case 1. $1_a \notin C_1(x)$ and $1_b \notin C_1(y)$. Then set $c(x) = 1_a$ and $c(y) = 1_b$.

Case 2. $C_1(x) = \{1_a, \alpha\}$ and $C_1(y) = \{1_a\}$, with $\alpha \in \{2_a, 2_b, 3\}$. Then set $c(y) = 1_b$. Suppose $c(x_1) = 1_a$ and $c(x_2) = \alpha$. The vertex x_1 has a neighbor colored 1_b (if not we would be in Case 1 by recoloring x_1). Let $x_{1,1}$ be the possible neighbor of x_1 not colored 1_b and β be its color. Recolor $x_{1,1}$ by its subsidiary color if $\alpha \neq \beta$ and $\alpha, \beta \in \{2_a, 2_b\}$. Then give a remaining color 2 to x.

Case 3. $C_1(x) = \{\alpha, \beta\}$ and $C_1(y) = \{1_a\}$, with $\alpha, \beta \in \{1_a, 1_b\}$. Then set $c(y) = 1_b$. A vertex among x_1 and x_2 has a neighbor of color 1_b (if not we would be in the first case by recoloring x_1 and x_2) and we suppose that this vertex is x_1. Two cases can occur, $C_1(x_2) = \{2_a, 2_b\}$ and consequently $C_2(x_2) = \{1_a, 1_b, 2_a, 2_b\}$ or $C_1(x_2) = \{\gamma, \delta\}$, with $\gamma \in \{1_a, 1_b\}$ and $\delta \in \{1_a, 1_b, 2_a, 2_b, 3\}$. In both cases, recolor every vertex at distance at most 3 from x colored 3 by its subsidiary color (by Property iii), such a vertex has no sibling colored 2) and set $c(x) = 3$.

Case 4. $C_1(x) = \{1_a, 1_b\}$ and $C_1(y) = \{2_a, 2_b\}$. Then set $c(y) = 1_b$. We suppose, without loss of generality, that $c(x_1) = 1_a$ and $c(x_2) = 1_b$. We have $1_b \in C_1(x_1)$ and $1_a \in C_1(x_2)$ (if not we would be in Case 1 by recoloring x_1 or x_2). Recolor every vertex at distance at most 3 from x colored 3 by its subsidiary color (by Property iii), such a vertex has no sibling colored 2) and set $c(x) = 3$.

Case 5. $C_1(x) = \{1_a, \alpha\}$ and $C_1(y) = \{1_a, \beta\}$, with $\alpha \in \{2_a, 2_b, 3\}$, $\beta \in \{2_a, 2_b, 3\}$. Then set $c(y) = 1_b$. Suppose $c(x_1) = 1_a$ and $c(x_2) = \alpha$, x_1 has a neighbor colored 1_b (if not we would be in Case by recoloring x_1) and $\{1_a, 1_b\} \in C_1(x_2)$, by Property i). If $\alpha = 3$, then change the color of x_2 by its subsidiary color α'. Let $\gamma \in \{2_a, 2_b\}$, with $\gamma \neq \alpha$, and $\gamma \neq \alpha'$ if $\alpha = 3$. Recolor every vertex at distance at most 2 from x colored γ by its subsidiary color and set $c(x) = \gamma$.

Case 6. $C_1(x) = \{1_a, 1_b\}$ and $C_1(y) = \{1_a, \alpha\}$, with $\alpha \in \{2_a, 2_b, 3\}$. Then set $c(y) = 1_b$. Suppose $c(x_1) = 1_a$ and $c(x_2) = 1_b$, x_1 has a neighbor colored 1_b (if not we would be in Case 1 by recoloring x_1) and x_2 has neighbors colored 1_a (if not we would be in Case 2 by recoloring x_2). Recolor every vertex at distance at most 2 from x colored 2_a by its subsidiary color and set $c(x) = 2_a$.

Case 7. $C_1(x) = \{1_a, 1_b\}$ and $C_1(y) = \{1_a, 1_b\}$. Suppose $c(x_1) = 1_a$ and $c(x_2) = 1_b$, x_1 has a neighbor colored 1_b and x_2 has a neighbor 1_a (if not we would be in Case 3 by recoloring x_1 or x_2). Recolor each neighbor of x_1 or x_2 colored 2_a by its subsidiary color (its sibling is colored 1) and set $c(x) = 2_a$. Recolor each neighbor of y_1 or y_2 colored 2_b by its subsidiary color (its sibling is colored 1) and set $c(y) = 2_b$.

Therefore, in each case, we obtain a $(1, 1, 2, 2, 3)$-coloring of G. □

The Petersen graph is an example of cubic graph which is not $(1, 1, 2, 3)$-colorable, showing that the result of Proposition 6 is tight in a certain sense. However, experiments suggest that the Petersen graph could be the only non $(1, 1, 2, 3)$-colorable subcubic graph, see Table 2.

Furthermore, the next result shows that the two colors 2 cannot be replaced by two colors 3 in the previous proposition.

Proposition 7. *There exist cubic graphs different from the Petersen graph that are not* $(1, 1, 3, 3, 3)$*-colorable.*

Proof. Consider the cubic graph depicted in Figure 7. Since it has diameter 3, hence no more than one vertex could be colored by a color 3. Moreover, it contains four triangles

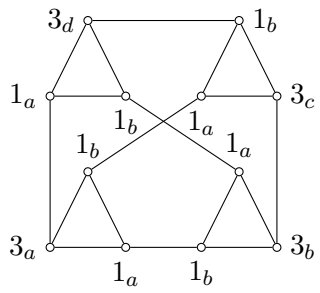

Figure 7: A cubic non $(1,1,3,3,3)$-colorable graph of order 12.

and each triangle should contain one vertex not colored 1. Thus, it is impossible to color it with the sequence $(1,1,3,3,3)$. □

$n\backslash S$	$(1,1)$	$(1,1,2)$	$(1,1,2,3)$	$(1,1,2,3,3)$
4	0	0	1	0
6	1	0	1	0
8	1	2	2	0
10	2	9	7	1
12	5	42	38	0
14	13	314	182	0
16	38	2808	1214	0
18	149	32766	8386	0
20	703	423338	86448	0
22	4132	6212201	1103114	0

Table 2: Number of S-chromatic cubic graphs of order n up to 22.

We now show that 3-irregular subcubic graphs are $(1,1,2)$-colorable. For subdivided graphs $S(G)$ of any graph G, note that $S(G)$ is $(1,1)$-colorable as it is bipartite.

Proposition 8. *Every 3-irregular subcubic graph is* $(1,1,2)$*-colorable.*

Proof. Let G be a 3-irregular graph and let $e = xy$ be any edge of G such that x and y both have degree at most 2. If no such edge exist then the graph is bipartite and consequently $(1,1)$-colorable.

Define a level ordering L_i, $0 \leq i \leq r = \epsilon(e)$, of (G,e).

We first construct a coloring c of the vertices of G from level r to 1 and with colors from the set $\{1_a, 1_b, 2\}$, that satisfies the following property :

i) No vertex of degree at most 2 is colored 2.

The set L_r induces a disjoint union of paths of order at most 3 in G. Since paths are $(1,1)$-colorable, L_r is $(1,1)$-colorable. Thus, Property i) is satisfied.

Assume that we have already colored all vertices of G of levels from r to $i+1$ and that we are going to color vertex $u \in L_i$, $1 \leq i \leq r-1$. If u has degree at most 2 then $\{1_a, 1_b\} \not\subseteq C_1(u)$. Hence u can be colored 1_a or 1_b and Property i) is satisfied. If u has degree 3 and if $C_1(u) \neq \{1_a, 1_b\}$, then u can be colored 1_a or 1_b. Else if $C_1(u) = \{1_a, 1_b\}$, let u_1 and u_2 be the colored neighbors of u, with $c(u_1) = 1_a$ and $c(u_2) = 1_b$. The vertex u_1 has a neighbor colored 1_b and the vertex u_2 has a neighbor colored 1_a, if not u_1 and u_2 could be recolored and u could be colored 1_a or 1_b. Thus, we have $C_1(u) \cup C_2(u) = \{1_a, 1_b\}$ and we can color u by the color 2.

Finally, it remains to color vertices of L_0, i.e. x and y. If $1_a \notin C_1(x)$ and $1_b \notin C_1(y)$ (or, symmetrically, $1_b \notin C_1(x)$ and $1_a \notin C_1(y)$). Then set $c(x) = 1_a$ and $c(y) = 1_b$ (or, symmetrically, $c(x) = 1_b$ and $c(y) = 1_a$). Let x_1 be the possible neighbor of x different from y and let y_1 be the possible neighbor of y different from x. Without loss of generality, suppose that $c(x_1) = 1_a$ and $c(y_1) = 1_a$. Suppose that x_1 has degree at most 2. If $C_1(x_1) = \{2\}$, then x_1 can be recolored 1_b and we can set $c(x) = 1_a$ and $c(y) = 1_b$. Else, $C_1(x_1) = \{1_b\}$ by Property i) and we can set $c(x) = 2$ and $c(y) = 1_b$. If x_1 has degree 3, then every colored neighbor of x has at most degree 2 and is colored 1_b by Property i). Thus, as $2 \notin C_1(x_1)$, we can set $c(x) = 2$ and $c(y) = 1_b$. Therefore, we obtain a $(1, 1, 2)$-coloring of G. □

4 $(1, 2, 3, \ldots)$-coloring

The question of whether cubics graphs have finite packing chromatic number or not was raised by Goddard et al. [11]. We give some partial results related to this question.

For the subdivision of a cubic graph, Proposition 1 along with Proposition 6 allow to obtain the following corollaries:

Corollary 3. *For every subcubic graph G, $S(G)$ is $(1, 3, 3, 5, 5, 7)$-colorable.*

Corollary 4. *For every subcubic graph G, $\chi_\rho(S(G)) \leq 6$.*

On the other side, it can be easily verified that $\chi_\rho(S(K_4)) = 5$.

For arbitrary cubic graphs, we can (only) state the following:

Proposition 9. *There exists a cubic graph with packing chromatic number* 13.

Proof. The cubic graph of order 38 and diameter 4 (which is the largest cubic graph with diameter 4) described independently in [1, 18] needs 13 colors to be packing colored (checked by computer). By running a brute force search algorithm, we found that at most 28 vertices can be colored with colors $\{1, 2, 3\}$. But, since this graph has diameter 4, then every color greater than 3 can be given to only one vertex, implying the use of all colors from $\{4, \ldots, 13\}$ to complete the coloring. □

The distribution of packing chromatic numbers for cubic graphs of order up to 20 is presented in Table 3. We also found, (with the help of a computer), a cubic graph of order 24 and packing chromatic number 11.

$n \setminus \chi_\rho$	4	5	6	7	8	9	10	11
4	1	0	0	0	0	0	0	0
6	1	1	0	0	0	0	0	0
8	0	3	2	0	0	0	0	0
10	0	3	15	1	0	0	0	0
12	0	7	42	36	0	0	0	0
14	0	13	252	222	22	0	0	0
16	0	34	907	2685	433	1	0	0
18	0	116	5277	21544	14050	314	0	0
20	0	151	22098	206334	226622	55284*		0

Table 3: Number of cubic graphs of order n with packing chromatic number χ_ρ up to 20.*There are 55284 cubic graphs of order 20 and with packing chromatic number between 9 and 10 (our program takes too long time to compute their packing chromatic numbers).

5 Concluding remarks

We conclude this paper by listing a few open problems:

- Is it true that any subcubic graph except the Petersen graph is $(1, 1, 2, 3)$-colorable?
- Is it true that any subcubic graph except the Petersen graph is $(1, 2, 2, 2, 2, 2)$-colorable?
- Does there exist a 3-irregular subcubic graph that is not $(1, 2, 2, 3)$-colorable?
- Is it true that any 3-irregular subcubic graph is $(1, 1, 3)$-colorable?
- Is it true that the subdivision of any subcubic graph is $(1, 2, 3, 4, 5)$-colorable?
- Does there exist a cubic graph with packing chromatic number larger than 13?

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