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Completely Independent Spanning Trees in Some Regular Networks

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Abstract

Let $k \geq 2$ be an integer and $T_1, \ldots, T_k$ be spanning trees of a graph $G$. If for any pair of vertices $(u, v)$ of $V(G)$, the paths from $u$ to $v$ in each $T_i$, $1 \leq i \leq k$, do not contain common edges and common vertices, except the vertices $u$ and $v$, then $T_1, \ldots, T_k$ are completely independent spanning trees in $G$. For $2k$-regular graphs which are $2k$-connected, such as the Cartesian product of a complete graph of order $2k - 1$ and a cycle and some Cartesian products of three cycles (for $k = 3$), the maximum number of completely independent spanning trees contained in these graphs is determined and it turns out that this maximum is not always $k$.

Keywords: Spanning tree, Cartesian product, Completely independent spanning tree.

1 Introduction

Let $k \geq 2$ be an integer and $T_1, \ldots, T_k$ be spanning trees in a graph $G$. The spanning trees $T_1, \ldots, T_k$ are edge-disjoint if $E(T_1) \cap \cdots \cap E(T_k) = \emptyset$. For a given tree $T$ and a given pair of vertices $(u, v)$ of $T$, let $P_T(u, v)$ be the set of vertices in the unique path between $u$ and $v$ in $T$. The spanning trees $T_1, \ldots, T_k$ are internally disjoint if for any pair of vertices $(u, v)$ of $V(G)$, $P_{T_1}(u, v) \cap \cdots \cap P_{T_k}(u, v) = \{u, v\}$. Finally, the spanning trees $T_1, \ldots, T_k$ are completely independent spanning trees if they are pairwise edge disjoint and internally disjoint.

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Disjoint spanning trees have been extensively studied as they are of practical interest for fault-tolerant broadcasting or load-balancing communication systems in interconnection networks: a spanning-tree is often used in various network operations; computing completely independent spanning-trees guarantees a continuity of service, as each can be immediately used as backup spanning tree if a node or link failure occurs on the current spanning tree. Thus, computing \( k \) completely independent spanning trees allows to handle up to \( k - 1 \) simultaneous independent node or link failures. In this context, a network is often modeled by a graph \( G \) in which the set of vertices \( V(G) \) corresponds to the nodes set and the set of edges \( E(G) \) to the set of direct links between nodes.

Completely independent spanning trees were introduced by T. Hasunuma [4] and then have been studied on different classes of graphs, such as underlying graphs of line graphs [4], maximal planar graphs [6], Cartesian product of two cycles [7] and complete graphs, complete bipartite and tripartite graphs [11]. Moreover, the decision problem that consists in determining if there exist two completely independent spanning trees in a graph \( G \) is NP-hard [6].

Other works on disjoint spanning trees include independent spanning trees which focus on finding spanning trees \( T_1, \ldots, T_k \) rooted at \( r \), such that for any vertex \( v \) the paths from \( r \) to \( v \) in \( T_1, \ldots, T_k \) are pairwise openly disjoint. the main difference is that \( T_1, \ldots, T_k \) are rooted at \( r \) and only the paths to \( r \) are considered. Thus \( T_1, \ldots, T_k \) may share common edges, which is not admissible with completely independent spanning trees. Independent spanning trees have been studied in several topologies, including product graphs [10], de Bruijn and Kautz digraphs [3, 5], and chordal rings [9]. Related works also include Edge-disjoint spanning trees, i.e. spanning-trees which are pairwise edge disjoint only. Edge-disjoint spanning trees have been studied on many classes of graphs, including hypercubes [1], Cartesian product of cycles [2] and Cartesian product of two graphs [8].

We use the following notations: for a tree, a vertex that is not a leaf is called an inner vertex. For a vertex \( u \) of a graph \( G \), let \( d_G(v) \) be its degree in \( G \), i.e. the number of edges of \( G \) incident with it.

For clarity, we recall the definition of the Cartesian product of two graphs: Given two graphs \( G \) and \( H \), the Cartesian product of \( G \) and \( H \), denoted \( G \square H \), is the graph with vertex set \( V(G) \times V(H) \) and edge set \( \{(u, u')(v, v')|(u = v \land u'v' \in E(H)) \lor (u' = v' \land u'v \in E(G))\} \).

The following theorem gives an alternative definition [4] of completely independent spanning trees.

**Theorem 1.1 ([4]).** Let \( k \geq 2 \) be an integer. \( T_1, \ldots, T_k \) are completely independent spanning trees in a graph \( G \) if and only if they are edge-disjoint spanning trees of \( G \) and for any \( v \in V(G) \), there is at most one \( T_i \) such that \( d_{T_i}(v) > 1 \).

It has been conjectured that in any \( 2k \)-connected graph, there are \( k \) completely independent spanning trees [6]. This conjecture has been refuted, as there exist \( 2k \)-connected graphs which do not contain two completely independent spanning trees [12], for any integer \( k \). However, the given counterexamples are not \( 2k \)-regular.
Proposition 1.2 ([12]). For any \( k \geq 2 \), there exist \( 2k \)-connected graphs that do not contain two completely independent spanning trees.

The proof of the previous proposition consists in constructing a \( 2k \)-connected graph with a large proportion of vertices of degree \( 2k \) adjacent to the same vertices and proving that these vertices of degree \( 2k \) can not be all adjacent to inner vertices in a fixed tree.

This article is organized as follows. Section 2 presents necessary conditions on \( 2r \)-regular graphs in order to have \( r \) completely independent spanning trees. Section 3 presents the maximum number of completely independent spanning trees in \( K_{n,m} \square C_n \), for \( n \geq 3 \) and \( m \geq 3 \). In particular, we exhibit the first \( 2r \)-regular graphs which are \( 2r \)-connected and which do not contain \( r \) completely independent spanning trees. In Section 4, we determine three completely independent spanning trees in some Cartesian products of three cycles \( C_{n_1} \square C_{n_2} \square C_{n_3} \), for \( 3 \leq n_1 \leq n_2 \leq n_3 \).

2 Necessary conditions on \( 2r \)-regular graphs

Proposition 2.1. If in a \( 2r \)-regular graph \( G \) there exist \( r \) completely independent spanning trees, then every spanning tree has maximum degree at most \( r+1 \).

Proof. By Theorem 1.1, every vertex should be of degree 1 in every spanning tree except in one spanning tree. Hence, in a spanning tree, a vertex is either of degree 1 (a leaf) or has degree between 2 and \( r+1 \) (an inner vertex), as \( 2r - (r-1) = r+1 \).

Let \( \text{IN}(T) \) be the set of inner vertices in a tree \( T \).

Proposition 2.2. If in a \( 2r \)-regular graph \( G \) of order \( n \) there exist \( r \) completely independent spanning trees, then there exists a spanning tree \( T \) among them such that \( |\text{IN}(T)| \leq \lceil n/r \rceil \).

Proof. Let \( T_1, \ldots, T_r \) be completely independent spanning trees in \( G \) and suppose that \( |\text{IN}(T_i)| > \lfloor n/r \rfloor \) for every \( i \in \{1, \ldots, r\} \). By Theorem 1.1, we have \( \sum_{i=1}^{r} |\text{IN}(T_i)| \leq n \). With our hypothesis, we have \( \sum_{i=1}^{r} |\text{IN}(T_i)| \geq (\lfloor n/r \rfloor + 1)r > n \), and a contradiction.

Proposition 2.3. If in a \( 2r \)-regular graph \( G \) of order \( n \) there exist \( r \) completely independent spanning trees \( T_1, \ldots, T_r \), then for every integer \( i, 1 \leq i \leq r \),

\[
\left\lceil \frac{n-2}{r} \right\rceil \leq |\text{IN}(T_i)| \leq n - \left\lfloor \frac{n-2}{r} \right\rfloor (r-1).
\]
Proof. In a spanning tree $T$ of a graph of order $n$ we recall that $\sum_{v \in V(T)} d_T(v) = 2n - 2$. By Proposition 2.1, we have $\sum_{v \in V(T)} d_T(v) \leq |\text{IN}(T)|r + n$ and we obtain $\lceil \frac{n-2}{r} \rceil \leq |\text{IN}(T)|$. By Theorem 1.1, $\sum_{i=1}^{r} |\text{IN}(T_i)| \leq n$. For a fixed integer $i$, using the previous inequality, we obtain $|\text{IN}(T_i)| \leq n - \lceil \frac{n-2}{r} \rceil(r - 1)$. \qed

**Definition 2.1.** Let $G$ be a $2r$-regular graph of order $n$ for which there exist $r$ completely independent spanning trees $T_1, \ldots, T_r$. A lost edge is an edge of $G$ that is in none of the spanning trees $T_1, \ldots, T_r$. We let $E^l$ be the set of lost edges, i.e. $E^l = E(G) - \bigcup_{1 \leq i \leq r} E(T_i)$. Let also $E^r_{T_i} = \{ uv \in E(G) | u, v \in \text{IN}(T_i), uv \notin E(T_i) \}$, for $i \in \{1, \ldots, r\}$, i.e. $E^r_{T_i}$ is the subset of edges of $E^l$ that have their two extremities in $\text{IN}(T_i)$.

**Proposition 2.4.** If in a $2r$-regular graph $G$ of order $n$ there exist $r$ completely independent spanning trees $T_1, \ldots, T_r$, then $|E^l| = r$.

**Proof.** We have $\sum_{i=1}^{r} |E(T_i)| + |E^l| = E(G) = rn$ and $\sum_{i=1}^{r} |E(T_i)| = r(n - 1)$. Hence, $|E^l| = r$. \qed

Since each edge of $E^r_{T_i}$ is also in $E^l$ and each edge of $E^l$ is in at most one set $E^r_{T_i}$ for some integer $i$, we have the following observation.

**Observation 2.5.** In a $2r$-regular graph $G$ of order $n$ for which there exist $r$ completely independent spanning trees $T_1, \ldots, T_r$, we have $\sum_{1 \leq i \leq r} |E^r_{T_i}| \leq |E^l| = r$.

**Definition 2.2.** The potential extra degree of a spanning tree $T$ in a $2r$-regular graph $G$ of order $n$ is $\text{ped}(T) = |\text{IN}(T)|r - n + 2$.

With Proposition 2.3, we have the following easy observation:

**Observation 2.6.** Let $G$ be a graph, for which there exist $r$ completely independent spanning trees $T_1, \ldots, T_r$. Then, for every $i$, $0 \leq i \leq r$, $\text{ped}(T_i) \geq 0$.

Note also that, by definition, the number of inner vertices of $T_i$ of degree at most $r$ is bounded by $\text{ped}(T_i)$.

**Proposition 2.7.** If in a $2r$-regular graph $G$ of order $n$ there exist $r$ completely independent spanning trees, then there exists a spanning tree $T$ among them such that $\text{ped}(T) \leq 2$ and $E^r_{T} \leq 1$, with strict inequalities if $r$ does not divide $n$.

**Proof.** By Proposition 2.2, there exists a tree $T$ among them such that $|\text{IN}(T)| \leq \lfloor n/r \rfloor$. Hence, $\text{ped}(T) \leq \lfloor n/r \rfloor r - n + 2 \leq 2$, with strict inequality if $r$ does not divide $n$. For every edge $uv$ in $E^r_{T}$, both $u$ and $v$ are adjacent to one inner vertex of every spanning tree other than $T$. Hence, both $u$ and $v$ have degree at most $r$ in $T$ and thus $\text{ped}(T) \geq 2|E^r_{T}|$. \qed
Note that the inequality \( \text{ped}(T) \geq 2|E_T^2| \) can be strict.

**Corollary 2.8.** Suppose that \( G \) is a \( 2r \)-regular graph of order \( n \) for which there exist \( r \) completely independent spanning trees \( T_1, \ldots, T_r \), for \( r \geq 3 \) and \( n \equiv 0 \) (mod \( r \)). Then, for every integer \( i \), \( 1 \leq i \leq r \), \( |IN(T_i)| = n/r \) and \( \text{ped}(T_i) = 2 \).

**Observation 2.9.** For a \( 2r \)-regular graph \( G \) of order \( n \) for which there exist \( r \) completely independent spanning trees \( T_1, \ldots, T_r \), for every tree \( T_i \), \( 1 \leq i \leq r \), and every edge \( e \) in \( E_{T_i}^2 \), the extremities of \( e \) have degree at most \( r \) in \( T_i \).

## 3 Cartesian product of a complete graph and a cycle

Let \( m \geq 3 \) and \( n \geq 2 \) be integers. In this section, the considered graphs are \( K_m \square P_n \) and \( K_m \square C_n \), \( m \geq 3 \).

Let \( V(K_m \square P_n) = V(K_m \square C_n) = \{ u_i^j, 0 \leq i \leq m-1, 0 \leq j \leq n-1 \} \) and \( E(K_m \square P_n) = \{ u_i^j u_k^j, 0 \leq i, k \leq m-1, i \neq k, 0 \leq j \leq n-1 \} \cup \{ u_i^j u_i^{j+1}, 0 \leq i \leq m-1, 0 \leq j \leq n-2 \} \). \( E(K_m \square C_n) = E(K_m \square P_n) \cup \{ u_i^j u_i^{j-1}, 0 \leq i \leq m-1 \} \).

For \( j \in \{0, \ldots, n-1\} \), the subgraphs \( K^j \) induced by \( \{ u_i^j, 0 \leq i \leq m-1 \} \) are thus complete graphs on \( n \) vertices that we call \( K \)-copies. In order to study the distribution of inner vertices of the spanning trees among the \( K \)-copies, we let \( V_j(T) = IN(T) \cap V(K^j) \) and \( n_j(T) = |V_j(T)| \) for any spanning tree \( T \) of \( K_m \square C_n \).

In the remaining, the subscript of \( u_i^j \) is considered modulo \( m \) and its super- and the subscripts of \( V_j(T) \) and \( n_j(T) \) are considered modulo \( n \).

**Proposition 3.1.** Let \( n \) and \( r \) be integers, \( n \geq 2 \), \( r \geq 2 \). There exist \( r \) completely independent spanning trees in \( K_r \square P_n \).

**Proof.** We construct \( r \) completely independent spanning trees \( T_1, \ldots, T_r \) as follows: \( E(T_i) = \{ u_{i-1}^j u_{i-1}^{j+1}, u_{i+1}^j u_{i+1}^{j+1} | j \in \{0, \ldots, n-2\} \} \cup \{ u_{i-1}^j u_{i-1}^{j-1}, u_{i+1}^j u_{i+1}^{j-1} | k \in \{0, \ldots, r-2\}, j \in \{0, \ldots, n-1\} \} \). \( \square \)

**Corollary 3.2.** Let \( n \) and \( r \) be integers, \( n \geq 3 \), \( r \geq 2 \). There exist \( r \) completely independent spanning trees in \( K_{2r} \square C_n \).

In the three next propositions, we will prove that there do not exist \( r \) completely independent spanning trees in \( K_{2r-1} \square C_n \), for some integers \( r \) and \( n \). Let \( p = |V(K_{2r-1} \square C_n)| = n(2r-1) \) and assume that there exist \( r \) completely independent spanning trees \( T_1, \ldots, T_r \) in \( K_{2r-1} \square C_n \). Let \( T \) be the spanning tree among them which minimizes \( |IN(T)| \), i.e. \( \text{ped}(T) \). By Proposition 2.2, \( T \) is such that \( |IN(T)| \leq |p/r| = 2n - [n/r], \text{ped}(T) \leq 2nr - [n/r]r - p + 2 \leq n - [n/r]r + 2 \leq 2 \) and \( |E_T^2| \leq 1 \). In order to establish this property we will consider all possible distributions of inner vertices of \( T \) among the different \( K \)-copies and prove that for each of them we have a contradiction.

The properties given in the following lemma will be useful.
Lemma 3.3. Let $a_i(T)$ be the number of $K$-copies which contains exactly $i$ inner vertices of $T$. The distribution of inner vertices among the different $K$-copies is such that:

i) if $n_j(T) \geq k$, for some integer $j$, then $|E_T^j| \geq \frac{1}{2}(k-1)(k-2)$;

ii) $n_j(T) < 4$, for every integer $j$;

iii) $a_3(T) \leq 1$;

iv) if $a_3(T) = 1$, then $n \equiv 0 \pmod{r}$ and $n \geq r$;

v) if $a_0(T) = 0$, then $a_3(T) \leq a_1(T) - [n/r]$; in particular $a_1(T) > a_3(T)$ and $a_1(T) \geq [n/r]$.

Proof. i) : A complete graph of order $k$ contains $\frac{1}{2}k(k-1)$ edges and only $k-1$ edges are in $E(T)$. Thus we have $|E_T^j| \geq \frac{1}{2}(k-1)(k-2)$.

ii) and iii) : If $n_j(T) \geq 4$ for some $j$ or $a_3(T) > 1$, then by i), we have $|E_T^j| \geq 2$. Hence, a contradiction.

iv) : As $\text{ped}(T) \leq n - [n/r]r + 2$, we have $|E_T^j| < 1$ in the case $n \not\equiv 0 \pmod{r}$.

v) : By ii), we have $|\text{IN}(T)| = a_1(T) + 2a_2(T) + 3a_3(T)$ and $a_2 = n - a_1(T) - a_3(T)$. Hence $|\text{IN}(T)| = a_1(T) + 2(n - a_1(T) - a_3(T)) + 3a_3(T) \leq 2n - [n/r]$ by the choice of $T$. Thus, $a_3(T) \leq a_1(T) - [n/r]$ and consequently $a_1(T) > a_3$ and $a_1(T) \geq [n/r]$.

We recall the following observation used in [12].

Observation 3.4 ([12]). If in a graph $G$ there exist $r$ completely independent spanning trees $T_1, \ldots, T_r$, then for every integer $i$, $1 \leq i \leq r$, every vertex is adjacent to an inner vertex of $T_i$.

Proposition 3.5. Let $n, r$ be integers, with $n \geq 3$ and $r \geq 6$. There do not exist $r$ completely independent spanning trees in $K_{2r-1} \square C_n$.

Proof. The proof is by contradiction, using Properties i)-v) of Lemma 3.3. Suppose that there exist $r$ completely independent spanning trees in $K_{2r-1} \square C_n$ and let $T$ be the tree from Proposition 2.2. If a $K$-copy $K^i$, $1 \leq i \leq n$, contains no inner vertex, then, by Observation 3.4, $n_{i-1}(T) + n_{i+1}(T) \geq 2r - 1 \geq 11$. Consequently, we have $n_{i-1}(T) \geq 6$ or $n_{i+1}(T) \geq 6$, contradicting Property ii). Hence $a_0(T) = 0$.

By Property v), $a_1(T) \geq [n/r] \geq 1$. Hence there exists an integer $i$, $0 \leq i \leq n - 1$, such that $n_i = 1$. Let $u$ be the (unique) vertex of $V_i(T)$. The vertex $u$ has degree at most $r+1$ in $T$ and is adjacent in $T$ to a vertex of $V_{i-1}(T) \cup V_{i+1}(T)$. Then, $u$ is adjacent in $T$ to at most $r$ vertices of $V(K^i)$. Thus, at least $r-2 \geq 4$ vertices are not adjacent in $T$ to $u$. Hence, these $r-2$ vertices are adjacent in $T$ to vertices of $V_{i-1}(T) \cup V_{i+1}(T)$ and consequently $n_{i-1}(T) + n_{i+1}(T) \geq 5$. Therefore, we have $n_{i-1}(T) \geq 3$ or $n_{i+1}(T) \geq 3$.

Assume, without loss of generality, that $n_{i+1}(T) \geq 3$. By Property ii), $n_{i+1}(T) = 3$ and by Property iii), $a_3(T) = 1$, i.e., $n_j(T) < 3$ for any $j \neq i$.  

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Hence, we have $n$ vertices are not adjacent in $T$. By Property v), $a_0 \leq n$ and $n$ are inner vertices and the dashed edge represents a lost edge.

Let $v$ be the (unique) vertex of $V_j(T)$. One vertex among $u$ and $v$ is adjacent in $T$ to two inner vertices (if not $T$ would be not connected). Suppose, without loss of generality, that $u$ is adjacent in $T$ to two inner vertices. Then $u$ is adjacent in $T$ to at most $r - 1$ vertices in $V(K^i)$. Thus, at least $r - 1 \geq 5$ vertices are not adjacent in $T$ to $u$. Therefore, at least 5 vertices are adjacent in $T$ to vertices of $V_{i-1}(T) \cup V_{i+1}(T)$ and consequently $n_{i-1}(T) + n_{i+1}(T) \geq 7$. Hence, we have $n_{i-1}(T) \geq 4$ or $n_{i+1}(T) \geq 4$, contradicting Property ii).

**Proposition 3.6.** Let $n, r$ be integers, with $4 \leq r \leq 5$ and $n \geq r + 1$. There do not exist $r$ completely independent spanning trees in $K_{2r-1} \square C_n$.

**Proof.** The proof is by contradiction, using Properties i)-v) of Lemma 3.3. Suppose that there exist $r$ completely independent spanning trees in $K_{2r-1} \square C_n$ and let $T$ be the tree from Proposition 2.2. If a $K$-copy $K^i$, $0 \leq i \leq n - 1$, contains no inner vertex, then $n_{i-1}(T) + n_{i+1}(T) \geq 7$. Consequently, we have $n_{i-1}(T) \geq 4$ or $n_{i+1}(T) \geq 4$, contradicting Property ii). Hence $a_0(T) = 0$.

By Property v), $a_1(T) \geq \lceil n/r \rceil \geq 2$. Thus, there exist two integers $i$ and $j$, $0 \leq i \leq j \leq n - 1$, such that $n_j(T) = n_i(T) = 1$, with $u \in V_i(T)$ and $v \in V_j(T)$.

First, suppose that $i = j - 1$. Each of $u$ and $v$ has degree at most $r + 1$ in $T$ and $u$ ($v$, respectively) is adjacent in $T$ to a vertex of $V_{i-1}(T) \cup V_{i+1}(T)$ (of $V_{j-1}(T) \cup V_{j+1}(T)$, respectively).

If $u$ and $v$ are adjacent in $T$, then one vertex among $u$ and $v$ is adjacent in $T$ to a vertex of $V_{i-1}(T) \cup V_{j+1}(T)$ (if not $T$ would be not connected). Suppose, without loss of generality, that $u$ is adjacent to two inner vertices. Then, at least $r - 1 \geq 3$ vertices of $V(K^i)$ are not adjacent in $T$ to $u$. Consequently, $n_{i-1}(T) \geq 4$ and we have a contradiction with Property ii).

Else if $u$ and $v$ are not adjacent in $T$, then both $u$ and $v$ are adjacent in $T$ to vertices of $V_{i-1}(T) \cup V_{j+1}(T)$ (if not, $T$ would be not connected). The vertices $u$ and $v$ are each adjacent in $T$ to at most $r$ vertices in $V(K^i) \cup V(K^j)$. Hence,
there remain at least $4r - 2 - 2r - 2 = 2r - 4 \geq 4$ vertices in $V(K^i) \cup V(K^j)$ that must be adjacent in $T$ to vertices of $V_{i-1}(T) \cup V_{j+1}(T)$ other than the neighbors of $u$ and of $v$. Consequently $n_{i-1}(T) + n_{j+1}(T) \geq 6$. Hence, we have $n_{i-1}(T) \geq 3$ and $n_{j+1}(T) \geq 3$, contradicting Property iii) or $n_{i-1}(T) \geq 4$ or $n_{j+1}(T) \geq 4$, contradicting Property ii).

Second, if $|i - j| > 1$, then one vertex among $u$ and $v$ is adjacent in $T$ to two inner vertices (if not $T$ would be not connected). Suppose, without loss of generality, that $u$ is adjacent to two inner vertices. At least $r - 1$ vertices of $V(K^i)$ are not adjacent in $T$ to $u$. Hence, if $r = 5$, we have $n_{i-1}(T) \geq 3$ and $n_{i+1}(T) \geq 3$, contradicting Property iii) or $n_{i-1}(T) \geq 4$ or $n_{i+1}(T) \geq 4$, contradicting Property ii). Consequently, we suppose that $r = 4$. Then, at least $r - 1 \geq 3$ vertices of $V(K^i)$ are not adjacent in $T$ to $u$. Therefore, we have $n_{i-1}(T) \geq 3$ or $n_{i+1}(T) \geq 3$.

Assume, without loss of generality, that $n_{i+1}(T) \geq 3$. By Property ii), $n_{i+1}(T) = 3$ and by Property iii), $a_3(T) = 1$, i.e., $n_j(T) < 3$ for any $j \neq i$. But, as $n > r$ and by Property v), $a_1 \geq 3$. Let $i'$ be such that $n_{i'}(T) = 1$, with $i' \neq i$ and $i' \neq i$. If $|i' - i| = 1$ or $|i' - j| = 1$, we have a contradiction, using the first point. Two vertices among $u$, $v$ and $u'$ should be adjacent to two inner vertices. Suppose it is the vertices $u$ and $v$. Using a similar argument than above, we obtain that $n_{j-1}(T) \geq 3$ or $n_{i+1}(T) \geq 3$. But, as $a_3(T) = 1$, the only possibility is to have $j = i + 2$, i.e. both $K$-copies with one internal vertices are adjacent to the same $K$-copy with three internal vertices.

In this case, as $r = 4$, then four vertices are not inner vertices in $V(K_{i+1})$, at least three vertices of $V(K^i)$ are not adjacent in $T$ to $u$ and at least three vertices of $V(K^j)$ are not adjacent in $T$ to $v$. Moreover, we have $n_{i-1}(T) \leq 2$ and $n_{j+1}(T) \leq 2$. Figure 1 illustrates this configuration. Thus, four vertices of $V(K_{i+1})$ are adjacent in $T$ to vertices of $V_{i+1}(T)$ and four vertices of $V(K^i) \cup V(K^j)$ are adjacent in $T$ to vertices of $V_{i+1}(T)$. However, by Observation 2.9, the vertices of $V_{i+1}(T)$ can be adjacent to at most seven leaves in $T$. Hence, we have a contradiction.

\[\square\]

**Proposition 3.7.** There do not exist five completely independent spanning trees in $K_9 \Box C_3$.

**Proof.** Suppose that there exist five completely independent spanning trees in $K_9 \Box C_3$ and let $T$ be the tree from Proposition 2.2. We recall that $|V(K_{9} \Box C_3)| = 27$ and $|\text{IN}(T)| \leq 6 - [3/4] = 5$. If a $K$-copy $K^i$, $0 \leq i \leq n - 1$, contains no inner vertex, then $n_{i-1}(T) \geq 5$ or $n_{i+1}(T) \geq 5$. Thus, we have a contradiction with Property ii). By property iv), as $n \neq 0 \pmod{r}$, we have $a_2(T) = 0$. Thus, the only possible distribution of inner vertices of $T$ is $a_1(T) = 1$ and $a_2(T) = 2$. Without loss of generality, suppose that $n_0(T) = 1$, $n_1(T) = 2$ and $n_2(T) = 2$, with $u \in V_1(T)$.

Let the position of a vertex $u_i^j$ be $i$. As $T$ should be connected, two pairs of inner vertices in different $K$-copies should be adjacent in $T$ among these five inner vertices. Thus, these five vertices have only three different positions. The
Figure 2: A pattern to have three completely independent spanning trees in \( K_3 \square C_n \), for \( n \equiv 0 \pmod{3} \).

vertex \( u \) has degree at most 6 in \( T \). Hence, there are \( r - 2 \geq 3 \) vertices of \( V(K^1) \) not adjacent in \( T \) to \( u \). As the inner vertices have only two positions different from the position of \( u \), it is impossible that every vertex is adjacent in \( T \) to an inner vertex of \( T \).

We now show positive results for the remaining values of \( r \) and \( n \). Some of the spanning trees were found using a computer to solve an ILP formulation of the problem.

**Proposition 3.8.** Let \( n \geq 3 \) be an integer such that \( n \equiv 0 \pmod{3} \). There exist three completely independent spanning trees in \( K_3 \square C_n \).

**Proof.** We construct three completely independent spanning trees \( T_1, T_2 \) and \( T_3 \) using repeatedly the pattern illustrated in Figure 2 on each three consecutive \( K \)-copies:

\[
E(T_1) = \{ u_0^j, u_0^{1+3j}, u_0^{2+3j}, u_0^{3+3j}, u_0, u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9, u_{10}, u_{11}, u_{12}, u_{13}, u_{14}, u_{15}, u_{16}, u_{17}, u_{18} \} 
\]

\( j \in \{0, \ldots, n/3 - 1\} \) - \( \{u_0^0, u_0^1\} \);

\[
E(T_2) = \{ u_0^j, u_0^{1+3j}, u_0^{2+3j}, u_0^{3+3j}, u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9, u_{10}, u_{11}, u_{12}, u_{13}, u_{14}, u_{15}, u_{16}, u_{17}, u_{18} \} 
\]

\( j \in \{0, \ldots, n/3 - 1\} \) - \( \{u_0^0, u_1^1\} \);

\[
E(T_3) = \{ u_0^j, u_0^{1+3j}, u_0^{2+3j}, u_0^{3+3j}, u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9, u_{10}, u_{11}, u_{12}, u_{13}, u_{14}, u_{15}, u_{16}, u_{17}, u_{18} \} 
\]

\( j \in \{0, \ldots, n/3 - 1\} \) - \( \{u_2^0, u_1^1\} \).

**Proposition 3.9.** Let \( n \geq 3 \) be an integer. There exist three completely independent spanning trees in \( K_5 \square C_n \).

**Proof.** By Proposition 3.8, there exist three completely independent spanning trees in \( K_5 \square C_n \), for \( n \equiv 0 \pmod{3} \). For \( n \equiv 1 \pmod{3} \), we use the pattern from Proposition 3.8 for \( K_3 \cup \ldots \cup K^{n-1} \), completed by the pieces of three completely independent spanning trees of \( K^0 \cup K^1 \cup K^2 \cup K^3 \) depicted in Figure 3 and whose edge sets are given in Appendix A.1. For \( n \equiv 2 \pmod{3} \), we use the pattern from Proposition 3.8 for \( K^5 \cup \ldots \cup K^{n-1} \), completed by the pieces of three
Proposition 3.10. There exist four completely independent spanning trees in $K_7 \Box C_3$.

Proof. The four completely independent spanning trees in $K_7 \Box C_3$ are depicted in Figure 5 and their edge sets are given in Appendix A.3.

Proposition 3.11. There exist four completely independent spanning trees in $K_7 \Box C_4$.

Proof. The four completely independent spanning trees in $K_7 \Box C_4$ are depicted in Figure 6 and their edge sets are given in Appendix A.4.

Proposition 3.12. There exist five completely independent spanning trees in $K_9 \Box C_4$.

Proof. The five completely independent spanning trees in $K_9 \Box C_4$ are depicted in Figure 7 and their edge sets are given in Appendix A.5.

Proposition 3.13. There exist five completely independent spanning trees in $K_9 \Box C_5$. 

Figure 3: The three completely independent spanning trees in $K_5 \Box C_n$, for $K^0 \cup K^1 \cup K^2 \cup K^3$ and $n \equiv 1 \pmod{3}$.

Figure 4: The three completely independent spanning trees in $K_5 \Box C_n$, for $K^0 \cup K^1 \cup K^2 \cup K^3 \cup K^4$ and for $n \equiv 2 \pmod{3}$.

completely independent spanning trees of $K^0 \cup K^1 \cup K^2 \cup K^3 \cup K^4$ depicted in Figure 4 and whose edge sets are given in Appendix A.2. Note that Figures 3 and 4 depict also three completely independent spanning trees in $K_5 \Box C_4$ and $K_5 \Box C_5$.

Proposition 3.10. There exist four completely independent spanning trees in $K_7 \Box C_3$.

Proof. The four completely independent spanning trees in $K_7 \Box C_3$ are depicted in Figure 5 and their edge sets are given in Appendix A.3.

Proposition 3.11. There exist four completely independent spanning trees in $K_7 \Box C_4$.

Proof. The four completely independent spanning trees in $K_7 \Box C_4$ are depicted in Figure 6 and their edge sets are given in Appendix A.4.

Proposition 3.12. There exist five completely independent spanning trees in $K_9 \Box C_4$.

Proof. The five completely independent spanning trees in $K_9 \Box C_4$ are depicted in Figure 7 and their edge sets are given in Appendix A.5.

Proposition 3.13. There exist five completely independent spanning trees in $K_9 \Box C_5$. 

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Proof. The five completely independent spanning trees in $K_9 \Box C_5$ are depicted in Figure 8 and their edge sets are given in Appendix A.6.

We end this section with a theorem summarizing the results for $K_m \Box C_n$.

Given a graph $G$, let $mcist(G)$ be the maximum integer $k$ such that there exist $k$ completely independent spanning trees in $G$.

**Theorem 3.14.** Let $m \geq 3$ and $n \geq 3$ be integers. We have:

$$mcist(K_m \Box C_n) = \begin{cases} \left\lceil \frac{m}{2} \right\rceil, & \text{if } (m = 3, 5) \lor (m = 7 \land n = 3, 4) \lor (m = 9 \land n = 4, 5)); \\ \left\lfloor \frac{m}{2} \right\rfloor, & \text{otherwise.} \end{cases}$$

**Proof.** For every even $m$, by Corollary 3.2, there exist $m/2$ completely independent spanning trees. Suppose $m$ is odd. For $m = 3$, Hasumuna and Morisaka [7] has proven that in any Cartesian product of 2-connected graphs, there are two completely independent spanning trees. By Propositions 3.12, 3.13, 3.10, 3.11 and 3.9, we obtain that there exist $\lceil m/2 \rceil$ completely independent spanning trees for $m = 5$ or $(m = 7 \land n = 3, 4)$ or $(m = 9 \land n = 4, 5)$.

In the other cases, by Propositions 3.5, 3.6, 3.7, there do not exist $\lfloor m/2 \rfloor$ completely independent spanning trees in these graphs. By Corollary 3.2, there exist $\lfloor m/2 \rfloor$ completely independent spanning trees in $K_{m-1} \Box C_n$. From these $\lfloor m/2 \rfloor$ completely independent spanning trees in $K_{m-1} \Box C_n$, we can construct $\lfloor m/2 \rfloor$ completely independent spanning trees in $K_m \Box C_n$. The graph $K_m \Box C_n$ contains $n$ vertices $u_0, \ldots, u_{n-1}$ not in $K_{m-1} \Box C_n$, with $u_j \in V(K^j)$ for $j =
Figure 7: Five completely independent spanning trees in $K_9 \square C_4$. 
Figure 8: Five completely independent spanning trees in $K_9 \bowtie C_5$. 
We require \( (T_M^{j}) \) to be a subgraph of it induced by the vertices \( (i, j, k) \) considered modulo \( n_1, n_2 \) and \( n_3 \), respectively.

By a level of \( TM(3, 3, q) \) we mean a subgraph of it induced by the vertices with the same third coordinate.

**Proposition 4.1.** Let \( p, p' \) and \( q \) be positive integers such that \( \gcd(p, p', q) = 1 \). There exist three completely independent spanning trees in \( TM(3p, 3p', 3q) \).

**Proof.** We define three completely independent spanning trees \( T_1, T_2 \) and \( T_3 \) in \( TM(3p, 3p', 3q) \) as follows: for \( j \in \{0, 1, 2\} \),

\[
E(T_{j-1}) = \{(i + j, j - i, i)(1 + i + j, -i + j, i), (i + j, j - i, i)(i + j, 1 - i + j, i), (i + j, j - i, i)(i + j, -1 + i + j, i), (i + j, j - i, i)(2 + i + j, j - i, i), (1 + i + j, j - i, i)(1 + i + j, j - i, i; \}
\]

\[
(1 + i + j, j - i, i)(1 + i + j, j - i, i), (1 + i + j, j - i, i)(1 + i + j, j - i, i), (1 + i + j, j - i, i)(1 + i + j, j - i, i), (1 + i + j, j - i, i)(1 + i + j, j - i, i), (1 + i + j, j - i, i)(1 + i + j, j - i, i), (i + j, -1 + i + j, i)(i + j, 1 - i + j, i), (i + j, 1 - i + j, i)(i + j, 1 - i + j, i)\}i \in \{0, \ldots, pp'q - 1\} - (j, j + 1, 0)\}j, j + 1, -1).}

We require \( \gcd(p, p', q) = 1 \), in order that \( T_1, T_2, T_3 \) contain every vertex of
Figure 10: The three completely independent spanning trees on the last four levels of $TM(3,3,q)$, for $q \equiv 1 \pmod{3}$ and $q > 2$.

$TM(3p,3p',3q)$, i.e. every edge is different for each value of $i$, $0 \leq i \leq pp'q - 1$. Figure 9 describes the pattern on three levels for these three spanning trees for $p = 1$ and $p' = 1$.

Proposition 4.2. For any integer $q \geq 3$, there exists three completely independent spanning trees in $TM(3,3,q)$.

Proof. First, if $q \equiv 0 \pmod{3}$, then Proposition 4.1 allows us to conclude. For $q \equiv 1 \pmod{3}$ ($q \equiv 2 \pmod{3}$, respectively), we define three completely independent spanning trees by using the pattern of Proposition 4.1 for every level except the last four (five, respectively) ones. If $q \equiv 1 \pmod{3}$, the trees are completed on the last four levels as depicted in Figure 10 (the corresponding edge sets are given in Appendix B.1). If $q \equiv 2 \pmod{3}$, the trees are completed on the last five levels as depicted in Figure 11 (the corresponding edge sets are given in Appendix B.2). □

5 Conclusion

We conclude this paper by listing a few open problems:

1. Determine conditions which ensure that there exist $r$ completely independent spanning trees in a graph.

2. Does any $2r$-connected graph with sufficiently large girth admit $r$ completely independent spanning trees?
Figure 11: The three completely independent spanning trees on the last five levels of $TM(3, 3, q)$, for $q \equiv 2 \pmod{3}$ and $q > 2$.

3. Is it true that in every 4-regular graph which is 4-connected, there exist 2 completely independent spanning trees?

4. Does the 6-dimensional hypercube $Q_6 = C_4 \square C_4 \square C_4$ admit 3 completely independent spanning trees?

References


A  Edge sets of the trees from Section 3

A.1 Three completely independent spanning trees in $K_5 \Box C_4$

$E(T_1) = \{u_0 u_1, u_0 u_2, u_0 u_3, u_0 u_4, u_1 u_4, u_1 u_2, u_2 u_4, u_2 u_3, u_3 u_1, u_3 u_2, u_4 u_3\}$

$E(T_2) = \{u_1 u_2, u_1 u_3, u_1 u_4, u_2 u_3, u_2 u_4, u_3 u_4, u_4 u_1, u_4 u_2, u_4 u_3\}$

$E(T_3) = \{u_2 u_3, u_2 u_4, u_2 u_1, u_3 u_2, u_3 u_4, u_4 u_3, u_4 u_1, u_4 u_2\}$

A.2 Three completely independent spanning trees in $K_5 \Box C_5$

$E(T_1) = \{u_0 u_3, u_0 u_4, u_0 u_2, u_0 u_1, u_1 u_4, u_1 u_3, u_1 u_2, u_2 u_3, u_2 u_1, u_3 u_4, u_3 u_2, u_4 u_1, u_4 u_3\}$

$E(T_2) = \{u_0 u_4, u_0 u_3, u_0 u_2, u_0 u_1, u_1 u_3, u_1 u_2, u_2 u_4, u_2 u_3, u_3 u_4, u_3 u_2, u_4 u_1, u_4 u_2\}$

$E(T_3) = \{u_0 u_2, u_0 u_4, u_0 u_3, u_0 u_1, u_1 u_4, u_1 u_3, u_1 u_2, u_2 u_4, u_2 u_3, u_3 u_4, u_3 u_2, u_4 u_1\}$

A.3 Four completely independent spanning trees in $K_7 \Box C_3$

$E(T_1) = \{u_0 u_3, u_0 u_4, u_0 u_2, u_0 u_1, u_1 u_2, u_1 u_3, u_1 u_4, u_2 u_1, u_2 u_3, u_2 u_4, u_3 u_1, u_3 u_4, u_4 u_1, u_4 u_2\}$

$E(T_2) = \{u_0 u_4, u_0 u_3, u_0 u_2, u_0 u_1, u_1 u_3, u_1 u_4, u_2 u_1, u_2 u_3, u_2 u_4, u_3 u_1, u_3 u_2, u_4 u_1, u_4 u_3\}$

$E(T_3) = \{u_0 u_2, u_0 u_4, u_0 u_3, u_0 u_1, u_1 u_2, u_1 u_4, u_2 u_1, u_2 u_3, u_2 u_4, u_3 u_1, u_3 u_2, u_4 u_3\}$

$E(T_4) = \{u_0 u_1, u_0 u_4, u_0 u_3, u_0 u_2, u_1 u_3, u_1 u_4, u_2 u_1, u_2 u_3, u_2 u_4, u_3 u_1, u_3 u_2, u_4 u_1\}$

A.4 Four completely independent spanning trees in $K_7 \Box C_4$

$E(T_1) = \{u_0 u_3, u_0 u_4, u_0 u_2, u_0 u_1, u_1 u_2, u_1 u_3, u_1 u_4, u_2 u_1, u_2 u_3, u_2 u_4, u_3 u_1, u_3 u_2, u_4 u_1, u_4 u_2\}$

$E(T_2) = \{u_0 u_4, u_0 u_3, u_0 u_2, u_0 u_1, u_1 u_3, u_1 u_4, u_2 u_1, u_2 u_3, u_2 u_4, u_3 u_1, u_3 u_2, u_4 u_3\}$

$E(T_3) = \{u_0 u_2, u_0 u_4, u_0 u_3, u_0 u_1, u_1 u_2, u_1 u_4, u_2 u_1, u_2 u_3, u_2 u_4, u_3 u_1, u_3 u_2, u_4 u_4\}$

$E(T_4) = \{u_0 u_1, u_0 u_4, u_0 u_3, u_0 u_2, u_1 u_3, u_1 u_4, u_2 u_1, u_2 u_3, u_2 u_4, u_3 u_1, u_3 u_2, u_4 u_1\}$

A.5 Five completely independent spanning trees in $K_9 \Box C_4$

$E(T_1) = \{u_0 u_3, u_0 u_4, u_0 u_2, u_0 u_1, u_1 u_2, u_1 u_3, u_1 u_4, u_2 u_1, u_2 u_3, u_2 u_4, u_3 u_1, u_3 u_2, u_4 u_1, u_4 u_2, u_4 u_3\}$

$E(T_2) = \{u_0 u_4, u_0 u_3, u_0 u_2, u_0 u_1, u_1 u_3, u_1 u_4, u_2 u_1, u_2 u_3, u_2 u_4, u_3 u_1, u_3 u_2, u_4 u_3\}$

$E(T_3) = \{u_0 u_2, u_0 u_4, u_0 u_3, u_0 u_1, u_1 u_2, u_1 u_4, u_2 u_1, u_2 u_3, u_2 u_4, u_3 u_1, u_3 u_2, u_4 u_4\}$

$E(T_4) = \{u_0 u_1, u_0 u_4, u_0 u_3, u_0 u_2, u_1 u_3, u_1 u_4, u_2 u_1, u_2 u_3, u_2 u_4, u_3 u_1, u_3 u_2, u_4 u_1\}$

$E(T_5) = \{u_0 u_3, u_0 u_4, u_0 u_2, u_0 u_1, u_1 u_2, u_1 u_3, u_1 u_4, u_2 u_1, u_2 u_3, u_2 u_4, u_3 u_1, u_3 u_2, u_4 u_3\}$
A.6 Five completely independent spanning trees in $K_3 \Box C_4$

$$E(T_1) = \{ u_0^0 u_2, u_0^0 u_3, u_0^0 u_4, u_0^0 u_5, u_0^0 u_6, u_0^0 u_7, u_0^0 u_8, u_1^1 u_2, u_1^1 u_3, u_1^1 u_4, u_1^1 u_5, u_1^1 u_6, u_1^1 u_7, u_1^1 u_8, u_2^2 u_2, u_2^2 u_3, u_2^2 u_4, u_2^2 u_5, u_2^2 u_6, u_2^2 u_7, u_2^2 u_8, u_3^3 u_2, u_3^3 u_3, u_3^3 u_4, u_3^3 u_5, u_3^3 u_6, u_3^3 u_7, u_3^3 u_8 \};$$

$$E(T_2) = \{ u_0^0 u_2, u_0^0 u_3, u_0^0 u_4, u_0^0 u_5, u_0^0 u_6, u_0^0 u_7, u_0^0 u_8, u_1^1 u_2, u_1^1 u_3, u_1^1 u_4, u_1^1 u_5, u_1^1 u_6, u_1^1 u_7, u_1^1 u_8, u_2^2 u_2, u_2^2 u_3, u_2^2 u_4, u_2^2 u_5, u_2^2 u_6, u_2^2 u_7, u_2^2 u_8, u_3^3 u_2, u_3^3 u_3, u_3^3 u_4, u_3^3 u_5, u_3^3 u_6, u_3^3 u_7, u_3^3 u_8 \};$$

$$E(T_3) = \{ u_0^0 u_2, u_0^0 u_3, u_0^0 u_4, u_0^0 u_5, u_0^0 u_6, u_0^0 u_7, u_0^0 u_8, u_1^1 u_2, u_1^1 u_3, u_1^1 u_4, u_1^1 u_5, u_1^1 u_6, u_1^1 u_7, u_1^1 u_8, u_2^2 u_2, u_2^2 u_3, u_2^2 u_4, u_2^2 u_5, u_2^2 u_6, u_2^2 u_7, u_2^2 u_8, u_3^3 u_2, u_3^3 u_3, u_3^3 u_4, u_3^3 u_5, u_3^3 u_6, u_3^3 u_7, u_3^3 u_8 \};$$

$$E(T_4) = \{ u_0^0 u_2, u_0^0 u_3, u_0^0 u_4, u_0^0 u_5, u_0^0 u_6, u_0^0 u_7, u_0^0 u_8, u_1^1 u_2, u_1^1 u_3, u_1^1 u_4, u_1^1 u_5, u_1^1 u_6, u_1^1 u_7, u_1^1 u_8, u_2^2 u_2, u_2^2 u_3, u_2^2 u_4, u_2^2 u_5, u_2^2 u_6, u_2^2 u_7, u_2^2 u_8, u_3^3 u_2, u_3^3 u_3, u_3^3 u_4, u_3^3 u_5, u_3^3 u_6, u_3^3 u_7, u_3^3 u_8 \};$$

$$E(T_5) = \{ u_0^0 u_2, u_0^0 u_3, u_0^0 u_4, u_0^0 u_5, u_0^0 u_6, u_0^0 u_7, u_0^0 u_8, u_1^1 u_2, u_1^1 u_3, u_1^1 u_4, u_1^1 u_5, u_1^1 u_6, u_1^1 u_7, u_1^1 u_8, u_2^2 u_2, u_2^2 u_3, u_2^2 u_4, u_2^2 u_5, u_2^2 u_6, u_2^2 u_7, u_2^2 u_8, u_3^3 u_2, u_3^3 u_3, u_3^3 u_4, u_3^3 u_5, u_3^3 u_6, u_3^3 u_7, u_3^3 u_8 \}.$$

B Edge sets of the trees from Section 4

B.1 Three completely independent spanning trees in the last four levels of $TM(3, 3, q)$

$$E(T_1) = \{(0,0,0)(1,0,0), (0,0,0)(0,1,0), (0,0,0)(2,0,0), (0,0,0)(0,0,1), (1,0,0)(1,2,0), (1,0,0)(2,0,0), (1,0,0)(1,0,1), (1,0,0)(1,1,0), (0,1,0)(1,1,1), (0,1,1)(1,1,0), (0,1,1)(0,1,0), (0,1,1)(1,0,0), (1,0,1)(2,0,1), (0,2,1)(1,2,0), (1,2,1)(2,0,1), (2,0,1)(1,0,2), (1,0,2)(2,0,0), (2,0,0)(1,0,1), (0,1,1)(0,1,0), (0,1,0)(1,0,0), (1,0,0)(1,1,0), (1,0,0)(2,0,0), (0,0,0)(0,0,1)\};$$

$$E(T_2) = \{(1,1,0)(2,1,0), (1,1,0)(1,1,0), (1,1,0)(1,2,0), (1,1,0)(1,1,1)\};$$

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\[(2,1,0)(0,1,0), (2,1,0)(2,0,0), (2,1,0)(2,1,1), (1,2,0)(2,0,0), (1,2,0)(1,2,1),\]
\[(0,0,1)(1,0,1), (0,0,1)(0,1,1), (0,0,1)(0,2,1), (1,0,1)(2,0,1), (1,0,1)(1,2,1),\]
\[(1,0,1)(1,0,2), (1,2,1)(2,1,2), (1,2,1)(2,0,2), (0,0,2)(2,0,2), (0,0,2)(2,0,2),\]
\[(0,0,2)(0,0,3), (2,0,2)(2,1,2), (2,0,2)(2,0,2), (2,0,2)(2,0,3), (2,0,2)(1,2,2),\]
\[(2,1,2)(1,1,2), (2,1,2)(2,1,3), (0,0,3)(1,0,3), (0,0,3)(2,2,3), (0,0,3)(0,0,4),\]
\[(0,2,3)(1,2,3), (0,2,3)(0,1,3), (0,2,3)(0,2,4), (1,2,3)(2,2,3), (1,2,3)(1,1,3),\]
\[(1,2,3)(1,2,4).\]

\[E(T_1) = \{(2,0,0)(0,0,0), (2,0,0)(2,0,0), (2,0,0)(2,0,1), (0,2,0)(1,2,0),\]
\[(0,2,0)(2,2,0), (0,0,2)(0,1,0), (0,0,2)(0,2,1), (2,2,0)(2,1,0), (2,0,2)(2,2,1),\]
\[(2,0,1)(0,0,1), (2,0,1)(2,1,1), (2,0,1)(2,0,2), (1,1,1)(2,1,1), (1,1,1)(1,0,1),\]
\[(1,1,1)(1,2,1), (2,1,1)(0,1,1), (2,1,1)(2,1,2), (0,1,2)(1,2,1), (0,1,2)(0,0,2),\]
\[(0,1,2)(0,2,2), (0,1,2)(0,1,3), (1,1,2)(1,0,2), (1,1,2)(1,1,3), (0,2,2)(1,2,2),\]
\[(1,2,2)(2,0,2), (2,0,2)(2,2,3), (1,0,3)(1,0,3), (1,0,3)(1,2,3), (1,0,3)(1,2,3),\]
\[(1,0,3)(1,0,4), (0,0,3)(0,0,3), (2,0,3)(2,2,3), (2,0,3)(2,0,4), (1,1,3)(2,1,3),\]
\[(1,1,3)(1,1,4).\]

\[E(T_2) = \{(2,0,0)(0,0,0), (2,0,0)(2,0,0), (2,0,0)(2,0,1), (0,2,0)(1,2,0),\]
\[(0,2,0)(2,2,0), (0,2,0)(2,0,1), (0,0,2)(2,2,1), (2,2,0)(2,1,0), (2,2,0)(2,2,1),\]
\[(0,1,1)(1,1,1), (0,1,1)(0,0,1), (0,1,1)(2,2,1), (0,1,1)(0,1,2), (1,1,1)(1,2,1),\]
\[(1,1,1)(1,0,1), (1,1,1)(1,1,2), (0,2,1)(2,1,2), (2,0,2)(1,0,2), (2,0,2)(2,1,2),\]
\[(2,0,2)(2,2,2), (2,0,2)(2,0,3), (2,0,2)(0,0,2), (0,2,2)(0,0,2), (0,2,2)(0,0,2),\]
\[(2,2,2)(2,2,2), (2,2,2)(2,0,3), (1,0,3)(1,0,3), (0,0,3)(0,0,3), (0,0,3)(0,0,3),\]
\[(0,0,3)(2,2,3), (2,0,3)(2,0,3), (2,0,3)(2,0,4), (2,1,3)(1,0,3), (2,1,3)(1,0,3),\]
\[(0,1,4)(0,1,4), (0,1,4)(0,1,4), (0,1,4)(0,1,4), (1,2,4)(1,2,4), (0,2,4)(1,2,4),\]
\[(0,2,4)(0,1,4), (0,2,4)(0,2,5), (1,2,4)(2,2,4), (1,2,4)(1,1,4), (1,2,4)(1,2,5),\]
\[E(T_3) = \{(2,0,0)(0,0,0), (2,0,0)(2,0,0), (2,0,0)(2,0,1), (0,2,0)(1,2,0),\]
\[(0,2,0)(2,2,0), (0,2,0)(2,0,1), (0,0,2)(2,2,1), (2,2,0)(2,1,0), (2,2,0)(2,2,1),\]
\[(0,1,1)(1,1,1), (0,1,1)(0,0,1), (0,1,1)(2,2,1), (0,1,1)(0,1,2), (1,1,1)(2,1,1),\]
\[(1,1,1)(1,0,1), (1,1,1)(1,1,2), (0,2,1)(2,1,2), (2,0,2)(1,0,2), (2,0,2)(2,1,2),\]
\[(2,0,2)(2,2,2), (2,0,2)(2,0,3), (2,0,2)(0,0,2), (0,2,2)(0,0,2), (0,2,2)(0,0,2),\]
\[(2,2,2)(2,2,2), (2,2,2)(2,0,3), (1,0,3)(1,1,3), (0,1,3)(0,0,3), (0,1,3)(0,0,3),\]
\[(0,1,3)(1,1,3), (1,1,3)(1,0,3), (1,1,3)(1,0,3), (0,2,3)(1,2,3), (0,2,3)(1,2,3),\]
\[(0,2,3)(0,2,4), (1,0,4)(2,2,4), (1,0,4)(1,1,4), (1,0,4)(2,2,4), (1,0,4)(1,0,5),\]
\[(2,0,4)(2,0,5), (2,0,4)(2,2,4), (2,0,4)(2,2,4), (2,0,4)(2,0,5), (1,1,4)(2,1,4), (1,1,4)(1,1,5).\]