A nonlinear derivative

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Outline

1 Introduction: 1st order derivative
   - On edge detection by derivative approach
   - Problem in the first order derivatives for image

2 A nonlinear elementary derivative

3 Properties of the nonlinear elementary derivative

4 Application to edge detection

5 Application to noise estimation

6 Perspectives and conclusion
1st derivative approach for edge detection

Considering edge detection by derivative approach, we can decompose the process into two steps:

- **regularization** (smoothing) for noise reduction, aliasing overcoming and detection of extended edge.
- gradient estimation thanks to derivatives (1st order) or 2nd order for zero-crossings

![Example of regularization by convolution](image1.png)

![Detection by elementary derivatives](image2.png)
Some nonlinear methods for regularization

Bilateral filtering

Non local means (NL-means)

PDE (Anisotropic)


and some derivatives

Gradient modulus obtained by the derivatives $[1 \ -1]$ and $[1 \ 0 \ -1]$:

Segmented edge images obtained with differentiating methods (no regul. in Roberts):

Is one of them satisfying?
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A 1st order derivative with no regularization can be defined as follows:

**Elementary/pure derivative**

\[ y^E_k = x_k - x_{k-1} \]

**A centered (regularizing) derivative**

\[ y^C_k = x_{k+1} - x_{k-1} \]

In the polynomial assumption (signal can be locally approximated by a polynomial function), accuracy of \( y^C_k / 2 \) is better than \( y^E_k \).
Asymmetrical and polarized discrete derivatives

Considering the previous derivative:

\[ y_k^C = x_{k+1} - x_{k-1} \]

\[ = (x_{k+1} - x_k) + (x_k - x_{k-1}) \]

\[ = T(x_{k+1} - x_k) - T[-(x_{k+1} - x_k)] \]

\[ + T(x_k - x_{k-1}) - T[-(x_k - x_{k-1})] \]

where \( T \) is the threshold operator: \( T(u) = u \) if \( u \geq 0 \), 0 elsewhere.

It can be rewrited as:

\[ y_k^C = y_k^{R+} + y_k^{R-} + y_k^{L+} + y_k^{L-} \]

where:

\[
\begin{align*}
y_k^{L+} &= T(x_k - x_{k-1}) \\
y_k^{L-} &= -T[-(x_k - x_{k-1})] \\
y_k^{R+} &= T(x_{k+1} - x_k) \\
y_k^{R-} &= -T[-(x_{k+1} - x_k)]
\end{align*}
\]
A nonlinear elementary derivative

**Definition of the nonlinear derivative (NLFS)**

\[ y^N_k = y^L_k + y^R_k = D^N_k x \]

with \( D^N_k \) nonlinear operator

Motivated by the asymmetry of edge profile in the discrete domain (center of symmetry does not correspond to a pixel):

Complementary version of the NLFS: \( y^O_k = y^{R+}_k + y^{L-}_k \).
A nonlinear elementary derivative
Response to Step and Peak signal

Signal

$y^C_k$
regularized

$y^N_k$
nonlinear, no reg.

$y^O_k$
nonlinear, no reg.

$y^E_k$
no reg.

complementary version
A nonlinear elementary derivative

Step signal transition defined by three points: the three typical responses

- **Aliasing**
- **Noise**
- **Closeness**

The Noise case and the Closeness one show that
\[D^N_k x \neq -D^N_k (-x)\]

- No smoothing effect (no spreading of the edge profiles): preserve edge profile
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It differentiates (sampled) continuous signals...

\[ MSE^V = (y - y^V)^2 \] and \[ MAX^V = \max |y - y^V| \]

Visually bad ... but efficient
It differentiates (sampled) continuous signals...

\( \text{MSE} \) measure on the \( \sin(2\pi k\alpha) \) function in respect of the sampling period \( \alpha \in [0; 0.5_{\text{Nyquist}}] \)

![Graph showing MSE curves for different sampling periods.](image)
We have $x_k = s_k + n_k$ with:

$s_k$: signal,

$n_k$: noise outcome of the random variable $N_k$

Assuming the signal is constant\(^1\)

$$y_k^N = T [n_k - n_{k-1}] - T [-(n_{k+1} - n_k)]$$

Developing the different cases:

\[
\begin{cases}
  y_k^N = n_{k+1} - n_{k-1} & \text{if } n_k > n_{k-1} \text{ and } n_k > n_{k+1} \quad (a) \\
  y_k^N = n_k - n_{k-1} & \text{if } n_k > n_{k-1} \text{ and } n_k \leq n_{k+1} \quad (b) \\
  y_k^N = n_{k+1} - n_k & \text{if } n_k \leq n_{k-1} \text{ and } n_k > n_{k+1} \quad (c) \\
  y_k^N = 0 & \text{if } n_k \leq n_{k-1} \text{ and } n_k \leq n_{k+1} \quad (d)
\end{cases}
\]

\(^1\)in nonlinear system, noise and signal cannot be studied separately
Introducing:

the pdf \( P_k \) for a continuous r.v. \( N_k \),
the r.v. \( Y_k \) for \( y_k^N \)

we deduce the pdf \( P_X(y) = y_k^N \):

\[
\begin{align*}
P_a(y) &= \int_{-\infty}^{\infty} P_k(x) dx \int_{-\infty}^{x} P_{k+1}(u) P_{k-1}(u - |y|) du \\
P_b(y) &= \int_{-\infty}^{\infty} P_{k+1}(x) dx \int_{-\infty}^{x} P_k(u) P_{k-1}(u - y) du \\
P_c(y) &= \int_{-\infty}^{\infty} P_{k-1}(x) dx \int_{x}^{\infty} P_{k+1}(u) P_k(u - y) du
\end{align*}
\]

and then (if \( P_{k-1} \equiv P_k \equiv P_{k+1} = \text{pdf } \text{GWN}(0, \sigma^2) \)):

\[
\begin{array}{c|ccc}
y_k & a & b & c \\
E\{Y_k\} & 0 & \frac{1}{2\sqrt{\pi}}\sigma & -\frac{1}{2\sqrt{\pi}}\sigma \\
E\{Y_k^2\} & \frac{\pi}{8}\sigma^2 & \frac{\pi}{16}\sigma^2 & \frac{\pi}{16}\sigma^2
\end{array}
\]
The two first centered moments of the response to GWN are:

\[
E\{Y_k\} = 0 \\
E\{Y_k^2\} = \frac{\pi}{4}\sigma^2 \approx 0.785\sigma^2.
\]

That means the variance \(\sigma^2\) of the original noise is reduced in the derivative to a variance \(0.785\sigma^2\) (centered noise).

**Remark**: The elementary linear derivative leads to a noise level \(2\sigma^2\).
Properties (summary) of the nonlinear derivative

- No smoothing $\rightarrow$ elementary derivative
- Symmetry $x_{-k} : D^N_k x_k = -D^N_k x_{-k}$
- Step signal: unique response and univocal localization
- No symmetry $-x : D^N_k x \neq -D^N_k (-x)$
- The signal is seen as a sum of positive and negative variations
- Noise reduction
- Can serve as elementary derivative
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Edge detection: localization

Notation for edge detection: gradient modulus $y_{i,j}^V = |G_{i,j}^V|$

**Image**

$y_i^C = \left[ \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right] / 4$

Regul. mask

$y_i^N$

$y_{ki,j}^E$
We define the noisy edge model:

\[ x_k = A \cdot H_k + n_k \]

where \( H_k \) is the Heaviside function and \( A \) the step amplitude. The step detection \((k = 0)\) is as follows:\(^2\):

\[ y_0 = T [A + n_0 - n_{-1})] - T [-n_1 + n_0] \]

This response corresponds to a (partial) random variable \( \mathcal{Y}_0 \). We deduce (example for \( GWN \)):

\[ E\{\mathcal{Y}_0^2\} = A^2 - A \cdot \frac{2}{\sqrt{\pi}} \cdot \sigma + 2\sigma^2 \quad \text{if} \quad \sigma \ll A \]

and in 2D:

\[ E\{\mathcal{Y}_0^2\} = A^2 - (A_x + A_y) \cdot \frac{2}{\sqrt{\pi}} \cdot \sigma + 4\sigma^2 \quad \text{if} \quad \sigma \ll A \]

\(^2\)more details can be found in O. Laligant, F. Truchetet, ”A Nonlinear Derivative Scheme Applied to Edge Detection,” IEEE TPAMI , vol.32, no.2, pp.242,257, Feb. 2010
Defining $SNR$ as:

$$SNR = \frac{\text{signal power average}}{\text{noise power average}} = \frac{\text{noisy signal power average} - \text{noise power average}}{\text{noise power average}}$$

The noise power average being $\frac{\pi}{2} \sigma^2$ in 2D, finally:

$$SNR_N \approx \frac{A^2}{\frac{\pi}{2} \sigma^2} \quad \text{if} \quad \sigma \ll A$$

$$Gain_{N/E} \approx \frac{8}{\pi}$$
Edge detection: elementary derivatives on real image

*House image*

Original image normalized to 1. $\sigma = 0.07$, threshold $t_d = 0.08$.

**Signal-to-noise ratio (GWN)**

For a large range of $\sigma$ and $t_d$, the measured gain in $SNR$ is $(SNR_N/\,SNR_E) \approx 2$. 
Edge detection: regularization

Regularization (smoothing) is motivated by noise reduction, aliasing overcoming and detection of extended edge.

With regularization, two nonlinear filtering schemes are possible:

If non-symmetrical smoothing filter ⇒
- two smoothing filters shifted by 1 pixel to preserve localization property (and symmetry)
- separation of the polarized detections
Example with a smoothing filter $H(z) = \frac{1}{6}z + \frac{2}{3} + \frac{1}{6}z^{-1}$:

$$y^C_{i,j} \quad y^{N-I}_{i,j} \quad y^{N-II}_{i,j}$$

$\text{SNR} \approx 4.8 \quad \text{SNR} \approx 7.3 \quad \text{SNR} \approx 7.0$

The benefit of the nonlinear schemes decreases as the smoothing increases.

The fine details are generally better detected with the NLFS II (regularization after the derivative operations).
The elementary nonlinear derivative and NLFS II do not detect thin lines (lighter, thickness 1 pixel).
Edge detection: other noise distributions

*Speckle noise*

*Salt noise*
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Contents of the noisy signal $x$:
- two elementary step signals at $k \in \{2, 11\}$
- two smoothed (aliased) steps at $k \in \{5, 8\}$ or two noisy steps
- two noise peaks at $k \in \{2, 11\}$

$y^+$ contains the noise components at $k \in \{2, 11\}$

Noise measures definition:

$$
\begin{align*}
    y^+ &= \min (y^{L+}, -y^{R-}) \\
    y^- &= -\min (-y^{L-}, y^{R+})
\end{align*}
$$
Introducing the pdf $P_k$ for a continuous and location-independent r.v. $N$ at $k$:

$$P_{Y^+}(y)_{y>0} = 2 \int_{-\infty}^{\infty} P_k(n)P_k(n-y)dn \int_{-\infty}^{n-y} P_k(x)dx$$

Choosing a moment '$p$' : $E[Y^{+p}] = \int_{-\infty}^{0} y^p P_{Y^+}(y)dy$

we deduce the corresponding noise estimator:

Noise estimator "NOLSE"

$$s^p = K_p \frac{1}{N} \sum_{k=1}^{N} y_{k}^{+p}$$

$K_p$ is deduced from the moment.

Example : $K_2 = \frac{8}{\pi}$ for the GWN, rate of convergence of the variance of the estimator: $O(N^{-1})$. 
Other methods for comparison

Compared to three other efficient methods

- **MAD** estimator [1]: the median value of the derivative:
  \[ s_{MAD}^2 = \left( \frac{\text{median}(|d|)}{0.6745} \right)^2 \]
- **FNVE** estimator [2]: an average of the image obtained by the difference of two versions of the Laplacian filter
- **TY** estimator [3]: a refinement of the previous method by discarding edges

### Noise estimation results

<table>
<thead>
<tr>
<th>$\sigma_{\text{added}}$</th>
<th>MAD</th>
<th>FNVE</th>
<th>TY</th>
<th>NOLSE$_2$</th>
<th>NOLSE$_d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>3.71</td>
<td>4.96</td>
<td>2.28</td>
<td>8.57</td>
<td>2.47</td>
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<td>2.00</td>
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<td>1.78</td>
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<td>5.00</td>
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<td>5.87</td>
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<td>4.63</td>
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<td>10.65</td>
<td>9.57</td>
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<tr>
<td>15.05</td>
<td>16.44</td>
<td>15.83</td>
<td>15.86</td>
<td>14.52</td>
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<tr>
<td>20.06</td>
<td>21.51</td>
<td>20.80</td>
<td>20.64</td>
<td>19.62</td>
<td>20.10</td>
</tr>
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<td>5.15</td>
<td>4.77</td>
<td>5.31</td>
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<tr>
<td>9.97</td>
<td>10.90</td>
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<td>15.19</td>
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<tr>
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<td>19.63</td>
<td>20.06</td>
</tr>
</tbody>
</table>

NOLSE$_2$ is the 2D extension of NOLSE.

NOLSE$_d$ is an estimator version based on a fitting of the measures distribution with the corresponding theoretical pdf.
Summary of noise estimation results

Best noise estimators (among four!):

Sensor noise: \(NOLSE_d, MAD\) (according to tests on sensors)

Additive \(GWN\): \(TY, NOLSE_d, NOLSE_2\)

Speckle (multiplicative) noise \(J = I + \sigma.\eta.I\) : \(NOLSE_2\)

Poisson noise \(P_{\lambda_k}(n) = \frac{\lambda^n e^{-\lambda}}{n!}\) : \(NOLSE_2\)

Salt-and-pepper noise : \(NOLSE_2\). Salt and Pepper can be measured separately.

Limitations of \(NOLSE_2\) and \(NOLSE_d\)

2D extension is two-directional: lines and textures in any direction can therefore be interpreted as noise.

The type of the distribution must be known to interpret the corresponding parameter (\(NOLSE_2\)).

Complex calculus (mainly for \(NOLSE_d\)) of the pdf and moments for most of the distribution types.
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Perspectives

2D definition, higher order

\[
\begin{align*}
F_{\\\\vee\, +} &= \frac{1}{3} \begin{bmatrix} 3 & -1 \\ -1 & -1 \end{bmatrix} & F_{\\\\vee\, -} &= \frac{1}{3} \begin{bmatrix} 1 & 1 \\ -1 & -3 \end{bmatrix} \\
F_{/\, +} &= \frac{1}{3} \begin{bmatrix} -1 & 3 \\ -1 & -1 \end{bmatrix} & F_{/\, -} &= \frac{1}{3} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}
\end{align*}
\]

multi-directionnal definition

- refinement in edge detection and noise estimation
- texture characterization
Conclusion

- A nonlinear **elementary** derivative leveraging discrete definition of signals
- Interest in localization, noise reduction and noise estimation
- $SNR$ gain of $3dB$ (at least) for edge detection with low computation cost

Further reading on this nonlinear derivative
- [algorithm available at](http://www.mathworks.com/matlabcentral/fileexchange/31029-edge-detection-by-nonlinear-derivatives)

Some connected references

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Thanks to the organizers

Thanks for your attention