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A characterization of b-chromatic and partial Grundy numbers by induced subgraphs

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Abstract

Gyárfás et al. and Zaker have proven that the Grundy number of a graph G satisfies $\Gamma(G) \geq t$ if and only if G contains an induced subgraph called a t -atom. The family of t -atoms has bounded order and contains a finite number of graphs. In this article, we introduce equivalents of t -atoms for b-coloring and partial Grundy coloring. This concept is used to prove that determining if $\varphi(G) \geq t$ and $\partial\Gamma(G) \geq t$ (under conditions for the b-coloring), for a graph G , is in XP with parameter t . We illustrate the utility of the concept of t -atoms by giving results on b-critical vertices and edges, on b-perfect graphs and on graphs of girth at least 7.

1 Introduction

Given a graph G , a *proper k -coloring* of G is a surjective function $c : V(G) \rightarrow \{1, \dots, k\}$ such that $c(u) \neq c(v)$ for any $uv \in E(G)$; the *color class* V_i is the set $\{u \in V \mid c(u) = i\}$ and a vertex v has *color* i if $v \in V_i$. A vertex v of color i is a *Grundy vertex* if it is adjacent to at least one vertex colored j , for every $j < i$. A *Grundy k -coloring* is a proper k -coloring such that every vertex is a Grundy vertex. The *Grundy number* of a graph G , denoted by $\Gamma(G)$, is the largest integer k such that there exists a Grundy k -coloring of G [11]. A *partial Grundy k -coloring* is a proper k -coloring such that every color class contains at least one Grundy vertex. The *partial Grundy number* of a graph G , denoted by $\partial\Gamma(G)$, is the largest integer k such that there exists a partial Grundy k -coloring of G . Let G and G' be two graphs. By $G \cup G'$ we denote the graph with vertex set $V(G) \cup V(G')$ and edge set $E(G) \cup E(G')$. Let $m(G)$ be the largest integer m such that G has at least m vertices of degree at least $m - 1$. A graph G is *tight* if it has exactly $m(G)$ vertices of degree $m(G) - 1$.



Figure 1: The graph $K_{3,3}^-$ with $\varphi(K_{3,3}^-) = 2$ (on the left) and $\varphi_r(K_{3,3}^-) = 3$ (on the right).

Another coloring parameter with domination constraints on the colors is the *b-chromatic number*. In a proper- k -coloring, a vertex v of color i is a *b-vertex* if v is adjacent to at least one vertex colored j , $1 \leq j \neq i \leq k$. A *b- k -coloring*, also called *b-coloring* when k is not specified, is a proper k -coloring such that every color class contains at least one b-vertex. The *b-chromatic number* of a graph G , denoted by $\varphi(G)$, is the largest integer k such that there exists a b- k -coloring of G . In this paper, we introduce the concept of *b-relaxed number*, denoted by $\varphi_r(G)$. A *b- k -relaxed coloring* of G is a b- k -coloring of a subgraph of G . The b-relaxed number of G is $\varphi_r(G) = \max_{H \subseteq G} (\varphi(H))$, for H an induced subgraph of G . Note that we have $\varphi(G) \leq \varphi_r(G) \leq \partial\Gamma(G)$. The difference between $\varphi(G)$ and $\varphi_r(G)$ can be arbitrary large. Let $K_{n,n}^-$ denotes the complete bipartite graph $K_{n,n}$ in which we remove $n - 1$ pairwise non incident edges (or $n - 1$ edges of a perfect matching in $K_{n,n}$) [1]. For this graph we have $\varphi(K_{n,n}^-) = 2$ and $\varphi_r(K_{n,n}^-) = n$ as Figure 1 illustrates it (for $n = 3$).

The concept of b-coloring has been introduced by Irving and Manlove [17], and a large number of papers was published (see e.g. [8, 20]). The b-chromatic number of regular graphs has been investigated in a serie of papers ([6, 18, 21, 23]). Determining the b-chromatic number of a tight graph is NP-hard even for a connected bipartite graph [19] and a tight chordal graph [13].

In this paper, we study the decision problems b-COL, b-r-COL and pG-COL with parameter t from Table 1.

	b-COL	b-r-COL	G-COL	pG-COL
Question	Does $\varphi(G) \geq t$?	Does $\varphi_r(G) \geq t$?	Does $\Gamma(G) \geq t$?	Does $\partial\Gamma(G) \geq t$?
Complexity class	undetermined	XP	XP [24]	XP

Table 1: The different decision problems with input a graph G and parameter t and their complexity class.

A decision problem is in FPT with parameter t if there exists an algorithm which resolves the problem in time $O(f(t) n^c)$, for an instance of size n , a computable function f and a constant c . A decision problem is in XP with parameter t if

there exists an algorithm which resolves the problem in time $O(f(t) n^{g(t)})$, for an instance of size n and two computable functions f and g .

The concept of t -atom was introduced independently by Gyárfás et al. [12] and by Zaker [24]. The family of t -atoms is finite and the presence of a t -atom can be determined in polynomial time for a fixed t . The following definition is slightly different from the definitions of Gyárfás et al. or Zaker, insisting more on the construction of every t -atom (some t -atoms can not be obtained with the initial construction of Zaker).

Definition 1.1 ([24]). *The family of t -atoms is denoted by \mathcal{A}_t^Γ , for $t \geq 1$, and is defined by induction. The family \mathcal{A}_1^Γ only contains K_1 . A graph G is in \mathcal{A}_{t+1}^Γ if there exists a graph G' in \mathcal{A}_t^Γ and an integer m , $m \leq |V(G')|$, such that G is composed of G' and an independent set I_m of order m , adding edges between G' and I_m such that every vertex in G' is connected to at least one vertex in I_m .*

Moreover, in the following sections, we say that a graph G in a family of graphs \mathcal{F} is *minimal*, if no graphs of \mathcal{F} is a proper induced subgraph of G . For example, a minimal t -atom A is a t -atom for which there are no t -atoms which are induced in A other than itself.

Theorem 1 ([12, 24]). *For a given graph G , $\Gamma(G) \geq t$ if and only if G contains an induced minimal t -atom.*

In this paper we prove equivalent theorems for b -relaxed number and partial Grundy number. In contrast with the minimal t -atoms, we can not define the minimal t -atoms for b -coloring as the smallest graphs such that G satisfies $\varphi(G) = t$ (also called b -critical graphs).

The paper is organized as follows: Section 2 is devoted to the definition of t -atoms for the partial Grundy coloring. This concept allows us to prove that the partial Grundy coloring problem is in XP with parameter t . Section 3 is similar to Section 2 but for b -relaxed-coloring. Section 4 is devoted to the concept of b -critical vertices and edges. Section 5 is about b -perfect graphs. Finally, Section 6 deals with graphs for which the b -relaxed and the b -chromatic numbers are equal.

2 Partial-Grundy- t -atoms: t -atoms for partial Grundy coloring

We start this section with the definition of t -atoms for partial Grundy coloring.

Definition 2.1. *Let t be an integer with $t \geq 2$. The family $\mathcal{A}^{\partial\Gamma}(t)$ of partial-Grundy- t -atoms (pG - t -atoms, for short) is obtained as follows: we begin by taking an independent set of t vertices, denoted by $C(t)$, called the center of the pG - t -atom and associating a different integer of $\{1, \dots, t\}$ to each vertex of $C(t)$. Let c_k be the vertex of $C(t)$ with associated integer k , $1 \leq k \leq t$. Let k be an integer with $1 \leq k \leq t$. A ∂ - k -dominating operation, on a graph G containing $C(t)$, consists in adding a set D of ℓ vertices, with $0 \leq \ell \leq t - k$ and*

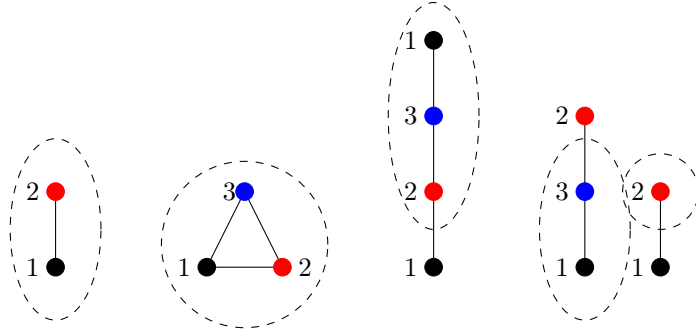


Figure 2: The minimal pG-2-atom (on the left) and the three minimal pG-3-atoms (the numbers are the colors of the vertices and the surrounded vertices form the centers).

associated integer k and adding edges between $G - \{c_k\}$ and $D \cup \{c_k\}$ such that every vertex of $\{c_{k+1}, \dots, c_t\}$ is adjacent to a vertex of $D \cup \{c_k\}$ and $D \cup \{c_k\}$ remains independent. Let $\mathcal{G}_1 = \{C(t)\}$. The family of graphs \mathcal{G}_i , for $2 \leq i \leq t$, is obtained from the graphs in \mathcal{G}_{i-1} by doing a ∂ - i -dominating operation on these graphs. Finally, set $\mathcal{A}^{\partial\Gamma}(t) = \mathcal{G}_t$.

Note that the associated integers induce a partial Grundy coloring of the pG- t -atom. These associated integers of the pG- t -atoms are not considered anymore when we use the notion of induced subgraph. Figure 2 illustrates several pG- t -atoms obtained using the previous construction.

Observation 2.2. For every graph G in $\mathcal{A}^{\partial\Gamma}(t)$, we have $|V(G)| \leq \frac{t(t-1)}{2}$.

Lemma 2.3. Let t and t' be two integers such that $1 \leq t' < t$. Every graph in $\mathcal{A}^{\partial\Gamma}(t)$ contains a graph in $\mathcal{A}^{\partial\Gamma}(t')$ as induced subgraph.

Proof. Every graph G in $\mathcal{A}^{\partial\Gamma}(t)$ contains a graph G' of $\mathcal{A}^{\partial\Gamma}(t')$: we can obtain G' by removing every vertex with an associated integer k in G , for $t' < k \leq t$, and by removing, afterwards, the vertices of G' not adjacent to the centers. \square

Note that the only minimal pG-2-atom is P_2 . The minimal pG-3-atoms are C_3 , P_4 and $P_2 \cup P_3$. These graphs are illustrated in Figure 2.

Theorem 2. Let G be a graph. For a graph G , we have $\partial\Gamma(G) \geq t$ if and only if G contains an induced minimal pG- t -atom.

Proof. Suppose that $\partial\Gamma(G) = t'$ with $t' \geq t$. Thus, there exists a partial Grundy coloring of G with t' colors. Let $u_1, \dots, u_{t'}$ be a set of Grundy vertices, each in a different color class of $V(G)$. The graph induced by $N[u_1] \cup \dots \cup N[u_{t'}]$ contains a pG- t' -atom. Hence, by Lemma 2.3, since G contains an induced pG- t' -atom, then it also contains an induced minimal pG- t -atom.

Suppose G contains an induced minimal pG- t -atom. Thus, the integers associated to the vertices correspond to a partial-Grundy coloring of this pG- t -atom. We can extend this coloring to a partial Grundy coloring of G with at least t colors, using a greedy algorithm [9] for the remaining vertices. \square

Proposition 2.4. *Let G be a graph of order n and let t be an integer. There exists an algorithm in time $O(n^{\frac{t(t-1)}{2}})$ to determine if $\partial\Gamma(G) \geq t$. Hence, the problem pG-COL with parameter t is in XP.*

Proof. By Theorem 2, it suffices to verify that G contains an induced minimal pG- t -atom to have $\partial\Gamma(G) \geq t$. Since the size of a minimal pG- t -atom is bounded by $\frac{t(t-1)}{2}$, we obtain an algorithm in time $O(n^{\frac{t(t-1)}{2}})$. \square

We finish this section by determining every graph G with $\partial\Gamma(G) = 2$.

Proposition 2.5. *For a graph G , we have $\partial\Gamma(G) = 2$ if and only if $G = K_{n,m}$, for some $n \geq 1$ and $m \geq 1$.*

Proof. Zaker [24] has proven that $\Gamma(G) = 2$ if and only if G is the disjoint union of copies of some $K_{n,m}$, for $n \geq 1$ and $m \geq 1$. Let n and m be positive integers. We can note that a graph containing two disjoint copies of $K_{n,m}$ contains an induced $P_2 \cup P_3$, hence a pG-3-atom. Hence, if $\partial\Gamma(G) = 2$, then $G = K_{n,m}$.

Moreover, $K_{n,m}$ does not contain induced C_3 , P_4 and $P_3 \cup P_2$. Hence, $\partial\Gamma(K_{n,m}) = 2$. \square

3 b- t -atoms: t -atoms for b-coloring

As in the previous section, we start this section with the definition of b- t -atoms (the notion of t -atom for b-coloring).

Definition 3.1. *Let t be an integer with $t \geq 2$. The family of b- t -atoms $\mathcal{A}^\varphi(t)$ is obtained as follows: we begin by taking an independent set of t vertices, denoted by $C(t)$, called the center of the b- t -atom and associating a different integer of $\{1, \dots, t\}$ to each vertex of $C(t)$. Let c_k be the vertex of $C(t)$ with associated integer k , $1 \leq k \leq t$. Let k be an integer with $1 \leq k \leq t$. A k -dominating operation on a graph G containing $C(t)$, consists in adding a set D of ℓ vertices, with $0 \leq \ell \leq t - 1$ and associated integer k and adding edges between $G - \{c_k\}$ and $D \cup \{c_k\}$ such that every vertex of $C(t) - \{c_k\}$ is adjacent to a vertex of $D \cup \{c_k\}$ and $D \cup \{c_k\}$ remains independent. Let $\mathcal{G}_1 = \{C(t)\}$. The family of graphs \mathcal{G}_i , for $2 \leq i \leq t$, is obtained from the graphs in \mathcal{G}_{i-1} by doing an i -dominating operation on these graphs. Finally, set $\mathcal{A}^\varphi(t) = \mathcal{G}_t$.*

For a fixed b- t -atom, the set of possible associated integers is called the set of b- t -atom colorings.

Note that the associated integers induce a b-coloring of the b- t -atom. These associated integers of the b- t -atom are not considered anymore when we use

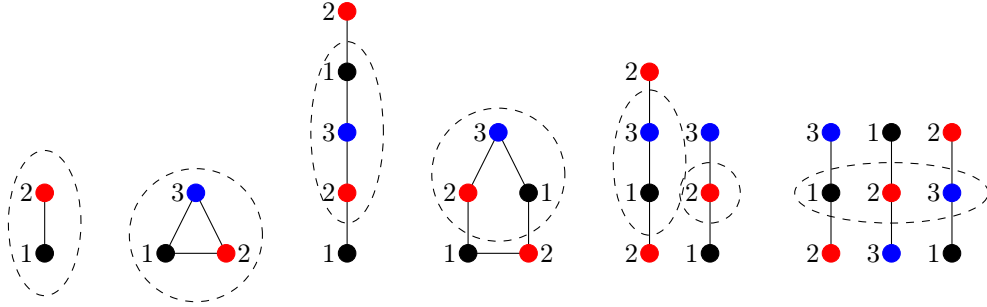


Figure 3: The minimal b-2-atom (on the left) and the five minimal b-3-atoms.

the notion of induced subgraph. Figure 3 illustrates several b- t -atoms obtained using the previous construction.

Observation 3.2. *For every graph G in $\mathcal{A}^\varphi(t)$, we have $|V(G)| \leq t^2$.*

Lemma 3.3. *Let t and t' be two integers such that $1 \leq t' < t$. Every graph in $\mathcal{A}^\varphi(t)$ contains a graph in $\mathcal{A}^\varphi(t')$ as induced subgraph.*

Proof. Every graph G in $\mathcal{A}^\varphi(t)$ contains a graph G' in $\mathcal{A}^\varphi(t')$: we can obtain G' by removing every vertex with an associated integer k in G , for $t' < k \leq t$, and by removing, afterwards, the vertices not adjacent to the centers. \square

Note that the only minimal b-2-atom is P_2 . The minimal b-3-atoms are C_3 , P_5 , C_5 , $P_3 \cup P_4$ and $P_3 \cup P_3 \cup P_3$. These graphs are illustrated in Figure 3.

Observation 3.4. *Every minimal pG- t -atom is an induced subgraph of a minimal b- t -atom or a minimal t -atom (an atom for the Grundy number).*

Proposition 3.5. *Let G be a graph. If $\varphi(G) \geq t$, then G contains an induced minimal b- t -atom.*

Proof. Suppose that $\varphi(G) = t'$, with $t' \geq t$. Thus, there exists a b-coloring of G with t' colors. Let $u_1, \dots, u_{t'}$ be a set of b-vertices, each in a different color class of $V(G)$. The graph induced by $N[u_1] \cup \dots \cup N[u_{t'}]$ contains a b- t' -atom. Hence, by Lemma 3.3, since G contains an induced b- t' -atom, then it also contains an induced minimal b- t -atom. \square

Theorem 3. *For a graph G , we have $\varphi_r(G) \geq t$ if and only if G contains an induced minimal b- t -atom.*

Proof. Suppose that the graph G contains an induced b- t -atom A . Since A admits, by definition, a b- t -coloring, we have $\varphi_r(G) \geq t$. Using Proposition 3.5, we obtain the reciprocal. \square

Definition 3.6. Let G be a graph. For an induced subgraph A of G , let $N(A) = \{v \in V(G) \setminus V(A) \mid uv \in E(G), u \in V(A)\}$. A b - t -atom A is feasible in G if there exists a b - t -coloring of $V(A)$ that can be extended to the vertices of $N(A)$ without using new colors.

Proposition 3.7. Let G be a graph. If G contains an induced feasible minimal b - t -atom and no induced feasible minimal b - t' -atom, for $t' > t$, then $\varphi(G) = t$.

Proof. If G contains no induced minimal b - t -atom, then by Proposition 3.5, we have $\varphi(G) < t$. Suppose that G contains an induced feasible minimal b - t -atom A and no b - t -coloring of G exists. We begin by considering that the vertices of $A \cup N(A)$ are already colored with t colors. We can note that, by assumption, no coloring of $A \cup N(A)$ (from the definition) can be extended to the whole graph using only t colors. Let t' be the largest integer such that the coloring can not be extended to a b - t' -coloring of the whole graph and let v be a vertex that can not be given a color among $\{1, \dots, t'\}$. Thus, we suppose that the coloring can be extended to a b - $(t' + 1)$ -coloring where v is colored by $t' + 1$. Since $A \cup N(A)$ is already colored, we have $v \in V(G) \setminus (A \cup N(A))$. The vertex v should be adjacent to vertices of every color, otherwise it could be colored. One vertex of each color class in $N(v)$ should be adjacent to vertices of each color class (except its color). Otherwise, the colors of the vertices of $N(v)$ could be changed in order v be colored. Let $N_2[v]$ be the set of vertices at distance at most 2 from v . The set $N_2[v]$ induces a graph containing a b - t' -atom where $N[v]$ contains the center of this b - $(t' + 1)$ -atom. Moreover, this b - $(t' + 1)$ -atom is feasible as the whole graph is b - $(t' + 1)$ -colorable, contradicting the hypothesis. \square

Proposition 3.8. Let G be a graph. If $\varphi(G) = t$, then G contains an induced feasible minimal b - t -atom and no induced feasible minimal b - t' -atom, for $t' > t$.

Proof. Suppose $\varphi(G) = t$. By Proposition 3.5, G contains an induced minimal b - t -atom. If no induced minimal b - t -atom is feasible, then there exists no b - t -coloring of G , a contradiction. \square

A direct consequence of Proposition 3.7 and Proposition 3.8 is the following.

Theorem 4. For a graph G , we have $\varphi(G) = t$ if and only if G contains an induced feasible minimal b - t -atom and no induced feasible minimal b - t' -atom, for $t' > t$.

Note that the problem of determining if a graph has a b - t -coloring is NP-complete even if t is fixed [22]. However, it does not imply that determining if $\varphi(G) \geq t$ for a graph G is NP-complete. In contrast with the b -chromatic number, determining if a graph has b -relaxed number at least t is in XP.

Proposition 3.9. Let G be a graph of order n and let t be an integer. There exists an algorithm in time $O(n^{t^2})$ to determine if $\varphi_r(G) \geq t$. In particular, the problem b - r -COL with parameter t is in XP.

Proof. It suffices to verify that G contains an induced minimal b - t -atom to determine if $\varphi_r(G) \geq t$. By Observation 3.2, the size of a minimal b - t -atom is bounded by t^2 . Thus, we obtain an algorithm in time $O(n^{t^2})$. \square

Another NP-complete problem for a graph G is to determine its b -spectrum [2], i.e. the set of integers k such that G is b - k -colorable. For a graph G satisfying $\varphi(G) = \varphi_r(G)$, our algorithm can be used as the two concepts became the same.

Corollary 3.10. *Let G be a graph of order n and t be an integer. There exists an algorithm in time $O(n^{t^2})$ to determine if $\varphi(G) \geq t$ for a graph G satisfying $\varphi(G) = \varphi_r(G)$.*

4 b -critical vertices and edges

The concept of b -critical vertices and b -critical edges has been introduced recently and since five years a large number of articles are considering this subject [1, 4, 5, 10, 25]. In this section, we illustrate how this notion is strongly connected with the concept of b - t -atom.

Definition 4.1 ([4, 10]). *Let G be a graph. A vertex v of G is b -critical if $\varphi(G - v) < \varphi(G)$. An edge e is b -critical if $\varphi(G - e) < \varphi(G)$. A vertex v (edge e , respectively) in a graph G is a b - t -trap, if there exists a b - t -atom of G that becomes feasible by removing v (e , respectively).*

Proposition 4.2. *Let G be a graph. A vertex v is b -critical if and only if it is in every feasible minimal b - $\varphi(G)$ -atom and v is not a b - $\varphi(G)$ -trap.*

Proof. Let $t = \varphi(G)$. First, if v is not in a feasible minimal b - t -atom, then $\varphi(G - v) = t$ and v is not b -critical. If v is a b - t -trap, then, by definition, $\varphi(G - v) = t$. Second, suppose v is not a b - t -trap. If v is in every feasible minimal b - t -atom, then, since every minimal b - t -atom in G does not contain any other feasible minimal b - t -atom as induced subgraph, $G - v$ does not contain a feasible minimal b - t -atom. Thus, v is b -critical. \square

Corollary 4.3. *If a graph G contains two induced feasible minimal b - $\varphi(G)$ -atoms with disjoint set of vertices, then it contains no b -critical vertex.*

Corollary 4.4. *Let G be a graph and v be a vertex of $V(G)$. If $\varphi(G - v) > \varphi(G)$, then G contains a minimal b - t -atom which is not feasible, for some $t > \varphi(G)$. If $\varphi(G - v) < \varphi(G) - 1$, then $G - v$ contains no feasible minimal b - t -atom, for $\varphi(G - v) < t \leq \varphi(G)$.*

In [1], Balakrishnan and Raj have proved the following theorem.

Theorem 5 ([1]). *Let G be a graph and v be a vertex of $V(G)$. We have $\varphi(G) - \lfloor \frac{|V(G)|}{2} \rfloor + 2 \leq \varphi(G - v) \leq \varphi(G) + \lfloor \frac{|V(G)|}{2} \rfloor - 2$.*

Moreover, some families of graphs which satisfy properties from Theorem 5 are determined. More precisely, they have determined the families of graphs for which there exists a vertex v such that $\varphi(G - v) = \varphi(G) - \lfloor \frac{|V(G)|}{2} \rfloor + 2$ or $\varphi(G - v) = \varphi(G) + \lfloor \frac{|V(G)|}{2} \rfloor - 2$. In contrast with the b-chromatic number, we have the following property about the b-relaxed number.

Proposition 4.5. *Let G be a graph. If a vertex v is b-critical, then $\varphi_r(G - v) = \varphi_r(G) - 1$.*

Proof. By Proposition 4.2, v is in every $b\text{-}\varphi(G)$ -atom. Let i be the integer associated to v in the construction of this $b\text{-}\varphi(G)$ -atom. By removing the vertices with associated integer i , we obtain a $b\text{-}(\varphi(G) - 1)$ -atom and thus $\varphi_r(G - v) = \varphi_r(G) - 1$. \square

Note that this proposition was already proved for trees [4] (a tree T satisfies $\varphi(T) = \varphi_r(T)$, since every b-coloring of a subtree can be extended to the whole tree).

Lemma 4.6. *Let G be a graph with $4 \leq |V(G)| \leq 5$ and $E(G) \neq \emptyset$. We have $\varphi_r(G - v) = \varphi_r(G) + \lfloor \frac{|V(G)|}{2} \rfloor - 2$, for every vertex v of $V(G)$, if and only if G contains two disjoint edges but no induced minimal b-3-atom.*

Proof. We can note that we have $\varphi_r(G - v) = \varphi_r(G) + \lfloor \frac{|V(G)|}{2} \rfloor - 2$ if and only if $\varphi_r(G - v) = \varphi_r(G)$.

First, if G contains no minimal b-3-atom and contains an edge, then $\varphi_r(G) = 2$. Moreover, if G contains two disjoint edges, then for any vertex v , $G - v$ contains P_2 and $\varphi_r(G - v) = 2$.

Second, suppose that for every vertex v , $\varphi_r(G - v) = \varphi_r(G)$. The only minimal b-3-atoms that contains at most five vertices are K_3 , C_5 and P_5 . Moreover, the only minimal b-4-atoms and b-5-atoms that contain at most five vertices are K_4 and K_5 . We are going to show that G is not one of these graphs

Case 1: $\varphi_r(G) = 5$. If G is a K_5 , then, by removing any vertex v , we obtain $\varphi_r(G - v) = 4$.

Case 2: $\varphi_r(G) = 4$. If G is a K_4 , then, by removing any vertex v , we obtain $\varphi_r(G - v) = 3$. If G contains an induced K_4 , $|V(G)| = 5$ and G is not K_5 , then there exists a vertex v such $G - v$ has no induced K_4 and $\varphi_r(G - v) = 3$.

Case 3: $\varphi_r(G) = 3$. If G contains an induced K_3 and no induced K_4 , then, since the induced K_3 in G have a common vertex v , we obtain $\varphi_r(G - v) = 2$. Moreover, if G is P_5 or C_5 , then, by removing any vertex v , we obtain $\varphi_r(G - v) = 2$.

Thus, we can suppose that $\varphi_r(G) = 2$. If G contains only edges with a common vertex v , then $\varphi_r(G - v) = 1$. Hence, G contains no b-3-atom and contain two disjoint edges. \square

The following theorem is a generalization of a conjecture of Blidia et al. [3] for the parameter φ_r . Note that the graphs P_4 , C_4 and $P_2 \cup P_2$ do not contain any induced minimal b-3-atom and contain two disjoint edges.

Theorem 6. *Let G be a graph. We have $\varphi_r(G - v) = \varphi_r(G) + \lfloor \frac{|V(G)|}{2} \rfloor - 2$, for every vertex v of $V(G)$, if and only one of these conditions is true about G :*

- i) G is P_2 or C_3 .
- ii) $E(G) = \emptyset$ and $4 \leq |V(G)| \leq 5$.
- iii) $4 \leq |V(G)| \leq 5$ and G contains two disjoint edges but no b-3-atom.

Proof. Note that if $|V(G)| \geq 6$, then, by Proposition 4.5, we can not have $\varphi_r(G - v) < \varphi_r(G) + \lfloor |V(G)|/2 \rfloor - 2$. Note also that if G contains only one vertex, then it can not satisfy $\varphi_r(G - v) < \varphi_r(G) + \lfloor |V(G)|/2 \rfloor - 2$.

First, if $2 \leq |V(G)| \leq 3$, then we have $\varphi_r(G - v) = \varphi_r(G) - 1$ if and only if G is a minimal b- t -atom. Hence, if and only if G is P_2 or C_3 . Second, if G contains no edges, then $\varphi_r(G) = 1$ and for any vertex v , $\varphi_r(G - v) = 1$. The third condition is obtained by Lemma 4.6. \square

Definition 4.7. *Let t be a positive integer and A be a b- t -atom. An edge e is b-atom-critical in A if $A - e$ is not a b- t -atom.*

Proposition 4.8. *Let G be a graph. An edge e is b-critical if and only if it is b-atom-critical in every feasible minimal b- $\varphi(G)$ -atom and e is not a b- $\varphi(G)$ -trap.*

Proof. Let $t = \varphi(G)$. First, if e is not b-atom-critical in a feasible minimal b- t -atom, then $G - e$ contains a feasible minimal b- t -atom and $\varphi(G - e) = t$. If e is a b- t -trap, then, by definition, $\varphi(G - e) = t$. Second, suppose that e is not a b- t -trap. If e is b-atom-critical in every feasible minimal b- t -atom, then, since every feasible minimal b- t -atom in G does not contain any other feasible minimal b- t -atom as subgraph in $G - e$, the graph $G - e$ does not contain a feasible minimal b- t -atom. Thus, e is b-critical. \square

Corollary 4.9. *If a graph G contains two induced feasible minimal b- $\varphi(G)$ -atoms with disjoint sets of b-atom-critical edges, then G contains no b-critical edge.*

5 b-perfect graphs

A b-perfect graph is a graph for which every induced subgraph satisfies that its b-chromatic number is equal to its chromatic number. More generally, we present the following definitions.

Definition 5.1 ([14]). *A graph G is b- χ - k -bounded, for k a positive integer, if $\varphi(G') - \chi(G') \leq k$, for every induced subgraph G' of G . A b- χ -0-bounded graph is called a b-perfect graph. A χ - k -unbounded b-atom, for k a positive integer,*

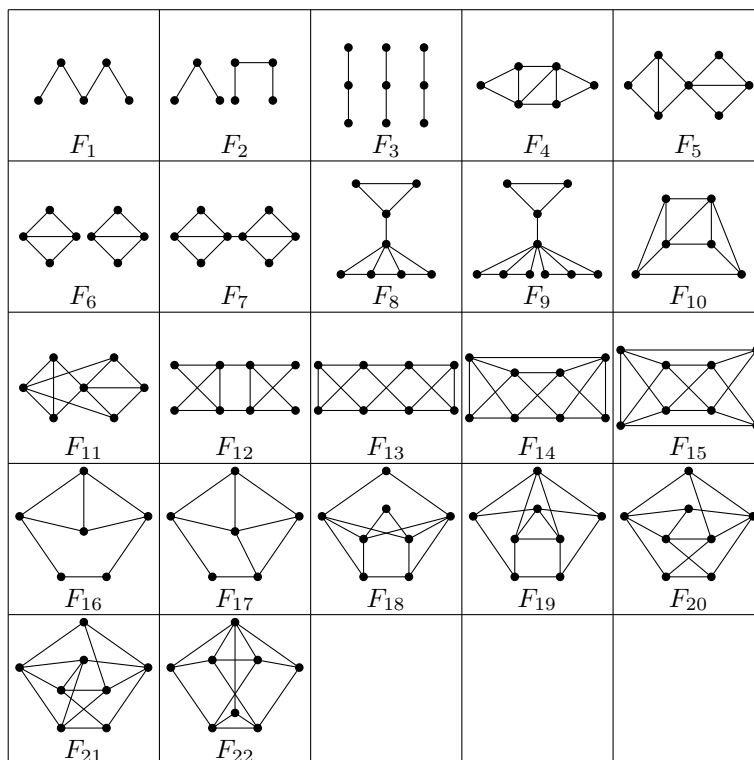


Figure 4: The family \mathcal{F} : the imperfect b-atoms [15].

is a b-atom which is not b - χ - k -bounded and which does not contain any b - χ - k -bounded induced minimal b-atom other than itself. An imperfect b-atom is a b-atom which is not b-perfect and which does not contain any induced minimal b-atom which is not b-perfect other than itself.

Hoang et al. [15] characterized b-perfect graphs by giving the family \mathcal{F} of forbidden induced subgraphs depicted in Figure 4. We recall the following theorem:

Theorem 7 ([15]). *A graph is b-perfect if and only if it contains no graph from \mathcal{F} as induced subgraph.*

Note that every graph in the family \mathcal{F} is a b- t -atom for some t . More precisely, F_1 , F_2 and F_3 are the only minimal bipartite b-3-atoms. The remaining graphs are minimal b-4-atoms that do not contain F_1 , F_2 and F_3 as induced subgraph and which admit a proper coloring with three colors (as mentioned in [16]). We can state the following property about b- t -atoms.

Observation 5.2. *Let t and t' be two integers with $t' < t$. If a minimal b- t -atom A contains an induced χ - k -unbounded b- t' -atom, then A is not b - χ - k -bounded.*

Theorem 8. *Let k be a positive integer. A graph G is not b - χ - k -bounded if and only if it contains a χ - k -unbounded b -atom.*

Proof. First, if G contains a χ - k -unbounded b -atom, then, by definition, G is not χ - k -bounded.

Second, suppose G is not b - χ - k -bounded. Then, there exists an induced subgraph A of G of minimal order which is not b - χ - k -bounded. By removing vertices of A we can only decrease the chromatic number. Thus, by removing vertices we can obtain a b - $\varphi(A)$ -atom which is χ - k -unbounded. \square

Corollary 5.3. *The graphs with b -chromatic number t which are b - χ - k -bounded, for fixed integers k and t , can be defined by forbidding a finite family of induced subgraphs: the χ - k -unbounded b -atoms. Hence, a graph G is b -perfect if and only if it does not contain imperfect b -atoms.*

Let b - χ -BOUNDED be the following decision problem with parameter $\varphi(G)$ and let k be an integer, with $0 \leq k < \varphi(G)$.

b - χ - k -BOUNDED

Instance : A graph G .

Parameter : $\varphi(G)$.

Question: Does $\varphi(G) - \chi(G) \geq k$?

By Corollary 5.3, we obtain the following corollary:

Corollary 5.4. *Let G be a graph and k be an integer, with $0 \leq k < \varphi(G)$. There exists an algorithm in time $O(n^{\varphi(G)^2})$ to solve b - χ - k -BOUNDED.*

Since a graph G is b -perfect if and only if it does not contain imperfect b -atoms, we have the following theorem:

Theorem 9. *The number of imperfect b -atoms is finite. A graph is an imperfect b -atom if and only if it is in the family \mathcal{F} (Figure 4).*

The previous theorem is a consequence of Theorem 7. By proving that every b -4-atom except K_4 contains an induced subgraph of the family \mathcal{F} and by using Proposition 5.2, we can possibly obtain another proof of Theorem 7.

6 b -chromatic and b -relaxed chromatic numbers

In this section we consider the b -relaxed number relatively to the b -chromatic number. About the relation between these parameters we can state the following properties.

Proposition 6.1. *For a graph G satisfying $\varphi(G) = \Delta(G) + 1$, we have $\varphi(G) = \varphi_r(G)$.*

Proof. Since every vertex is adjacent to at most $\Delta(G)$ neighbors, it is always possible to extend a $(\Delta + 1)$ -coloring to a new vertex. \square

Proposition 6.2. For a tree T , we have $\varphi(T) = \varphi_r(T)$.

Proof. It is always possible to extend a k -coloring to a new vertex in a tree. \square

We finish this paper by proving that when a graph G has sufficiently large girth, we have $\varphi(G) = \varphi_r(G)$.

Theorem 10. Let G be a graph with girth g and $\varphi_r(G) \geq 3$. If $g \geq 7$, then $\varphi(G) = \varphi_r(G)$.

Proof. Suppose that $\varphi_r(G) = t$. Let A_t be a colored b - t -atom of G and let c_1, \dots, c_t be the vertices of the center of this b - t -atom. Moreover, we suppose that c_i is colored by i . If A_t is not feasible, then there exists a subset of vertices in $N(A_t)$ such that this coloring can not be extended to these vertices. By assumption, we suppose that A_t is not feasible. For a vertex $u \in N(A_t)$, we denote by $N_c(u)$ the set $\{i \in \{1, \dots, t\} \mid \exists v \in N(u) \cap N[c_i]\}$. We denote by $N[A_t]$, the set of vertices $A_t \cup N(A_t)$. In order to have a contradiction we begin by proving the following properties:

- i) if $u \in N(A_t)$, then u is not adjacent to two vertices of $N[c_i]$, for $1 \leq i \leq t$;
 - ii) if $u, v \in N(A_t)$ and $N_c(u) \cap N_c(v) \neq \emptyset$, then u and v are at distance at least 3;
 - iii) if $u, v \in N[c_i]$ and $u', v' \in N[c_i]$ for some i , $1 \leq i \leq t$, then no two pairs of vertices among $\{u, v, u', v'\}$ are adjacent.
- i) If u is adjacent to two vertices of $N[c_i]$, for some i , $1 \leq i \leq t$, then u is in a cycle of length at most 4.
 - ii) If two vertices u and v from $N(A_t)$ are at distance at most 2 and are both adjacent to vertices of $N[c_i]$, for $1 \leq i \leq t$, then u and v are in a cycle of length at most 6.
 - iii) If two pairs of vertices among $\{u, v, u', v'\}$ are adjacent then there is a cycle of length at most 6 in G .

We are going to prove that either each vertex $u \in N(A_t)$ can be given a color from $\{1, \dots, t\}$ or the graph contains a b - $(t+1)$ -atom (which contradicts $\varphi_r(G) = t$). First, we will consider the vertices from $N(A_t)$ which are not only adjacent to centers. Afterward, we will consider the remaining vertices of $N(A_t)$.

Case 1: u has at least one neighbor in A_t which is not a center.

By properties i) and ii), u has at most t neighbors in $N[A_t]$. If u has less than t neighbors in $N[A_t]$, then it can be given a color, hence assume u has exactly t neighbors in $N[A_t]$. If u has less than t neighbors in A_t , then there exists a neighbor w of u in $N(A_t)$. By property ii), $|N_c(w)| = 1$, i.e. w has two neighbors in $N[A_t]$ and thus can be recolored in order to give its original color to u . If u has exactly t neighbors in A_t , let v_1, \dots, v_ℓ be

the neighbors of u in A_t which are not in the center. Let c' be the color of v_1 and let c'' be the color of v_2 if it exists. Moreover, let i_1 and i_2 be two integers such that $v_1 \in N(c_{i_1})$ and $v_2 \in N(c_{i_2})$ (if $\ell \geq 2$).

Subcase 1.a: suppose that $\ell = 1$, i.e., u is adjacent to $t - 1$ vertices from the center and to one vertex in A_t which is not a center.

We can note that no vertex from $N[c_{i_1}]$ can have a neighbor in A_t , since it would create a cycle of length at most 6. Consequently, we can exchange the color of v_1 with the color of one vertex from $N(c_{i_1})$ and color u by c' .

Subcase 1.b: suppose that $\ell \geq 2$, i.e., u is adjacent to at most $t - 2$ vertices from the center and to at least two vertices in A_t which are not centers.

The vertex v_1 has no neighbor in $A_t - N[c_{i_1}]$ and v_2 has no neighbor in $A_t - N[c_{i_2}]$ since it would create a cycle of length at most 6. If a vertex of $N(c_{i_1}) - \{v_1\}$ has no neighbor of color c' , then we exchange the color of v_1 with the color of this vertex and color u by c' . If a vertex of $N(c_{i_2}) - \{v_2\}$ does not have a neighbor of color c'' , then we exchange the color of v_2 with the color of this vertex and color u by c'' . Thus, we suppose that every vertex of $N(c_{i_1}) - \{v_1\}$ has a neighbor of color c' and every vertex of $N(c_{i_2}) - \{v_2\}$ has a neighbor of color c'' .

In the case $c'' = i_1$ and $c' = i_2$, by property iii), no vertex of $N[c_{i_1}] \cup N[c_{i_2}]$ has a neighbor in $A_t - N[c_{i_1}] \cup N[c_{i_2}]$ other than these vertices of color c' and c'' . However, there could exist two adjacent vertices w and w' with $w \in N[c_{i_1}]$ and $w' \in N[c_{i_2}]$. If $t = 3$, then w and w' are colored by the same color and thus can not be adjacent. Consequently, we can exchange the color of v_1 with the color of v_2 , the color of c_{i_1} with the color of c_{i_2} and afterward we can exchange the color of one vertex from $N(c_{i_1}) - \{w, v_1\}$ (if $t = 3$, then we take w) with the color of v_1 and color u by c'' .

In the case $c'' \neq i_1$ (or $c' \neq i_2$, respectively), we suppose, without loss of generality, that $c' = i_2$ (or $c'' = i_1$, respectively). In this case, we exchange the color of c_{i_1} with the color of c_{i_2} and the color of v_1 with the color of a vertex w of color i_1 in $N(c_{i_2})$ (the color of v_2 with the color of a vertex w of color i_2 in $N(c_{i_1})$, respectively) and color u by c' (by c'' , respectively). We can note that the vertex w in $N(c_{i_2})$ (the vertex w in $N(c_{i_1})$, respectively) can be adjacent to a vertex of color c' (of color c'' , respectively). However, if w is adjacent to a vertex of color c' (of color c'' , respectively), then no vertex of $N(c_{i_2}) - \{w, v_2\}$ (of $N(c_{i_1}) - \{w, v_1\}$, respectively) is adjacent to a vertex of color c' (of color c'' , respectively). If w is adjacent to a vertex of this color, then we exchange the color of w with the color of a vertex of $N(c_{i_2}) - \{w, v_2\}$ (of $N(c_{i_1}) - \{w, v_1\}$, respectively). We can remark that if $t = 3$, then w can not be adjacent to a vertex of color c' (of color c'' , respectively) and the previous case never happen.

Case 2: every neighbor of u is in the center.

If u does not have neighbors in each color class, then we can give a color to u . The vertices from $N(u) \cap A_t$ are centers, and thus these vertices

have neighbors from each color class. If every vertex v in $N(u) \cap N(A_t)$ colored by a given color c does not have neighbors from each color class, then we recolor these vertices and give the color c to u . If u remains uncolorable, the set $N_2[u]$ induces a graph containing a $b-(t+1)$ -atom, which contradicts $\varphi_r(G) = t$.

□

We think that the previous theorem can be used to determine the family of graphs of girth at least 7 satisfying $\varphi(G) = m(G)$. It has already been proven that graphs of girth at least 7 have b -chromatic number at least $m(G) - 1$ [7].

Corollary 6.3. *Let G be a graph of girth at least 7 and of order n and let t be an integer. There exists an algorithm in time $O(n^{t^2})$ to determine if $\varphi(G) \geq t$.*

7 Open questions

We conclude this article by listing few open questions.

1. For which family of graphs are the b -relaxed number and the b -chromatic number equal?
2. Does there exists an easy characterization of feasible b - t -atoms?
3. Does there exists an FPT algorithm, with parameter t , to determine if $\varphi(G) \geq t$?

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