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# A characterization of b-chromatic and partial Grundy numbers by induced subgraphs

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## Abstract

Gyárfás et al. and Zaker have proven that the Grundy number of a graph  $G$  satisfies  $\Gamma(G) \geq t$  if and only if  $G$  contains an induced subgraph called a  $t$ -atom. The family of  $t$ -atoms has bounded order and contains a finite number of graphs. In this article, we introduce equivalents of  $t$ -atoms for b-coloring and partial Grundy coloring. This concept is used to prove that determining if  $\varphi(G) \geq t$  and  $\partial\Gamma(G) \geq t$  (under conditions for the b-coloring), for a graph  $G$ , is in XP with parameter  $t$ . We illustrate the utility of the concept of  $t$ -atoms by giving results on b-critical vertices and edges, on b-perfect graphs and on graphs of girth at least 7.

## 1 Introduction

Given a graph  $G$ , a *proper  $k$ -coloring* of  $G$  is a surjective function  $c : V(G) \rightarrow \{1, \dots, k\}$  such that  $c(u) \neq c(v)$  for any  $uv \in E(G)$ ; the *color class*  $V_i$  is the set  $\{u \in V \mid c(u) = i\}$  and a vertex  $v$  has *color*  $i$  if  $v \in V_i$ . A vertex  $v$  of color  $i$  is a *Grundy vertex* if it is adjacent to at least one vertex colored  $j$ , for every  $j < i$ . A *Grundy  $k$ -coloring* is a proper  $k$ -coloring such that every vertex is a Grundy vertex. The *Grundy number* of a graph  $G$ , denoted by  $\Gamma(G)$ , is the largest integer  $k$  such that there exists a Grundy  $k$ -coloring of  $G$  [11]. A *partial Grundy  $k$ -coloring* is a proper  $k$ -coloring such that every color class contains at least one Grundy vertex. The *partial Grundy number* of a graph  $G$ , denoted by  $\partial\Gamma(G)$ , is the largest integer  $k$  such that there exists a partial Grundy  $k$ -coloring of  $G$ . Let  $G$  and  $G'$  be two graphs. By  $G \cup G'$  we denote the graph with vertex set  $V(G) \cup V(G')$  and edge set  $E(G) \cup E(G')$ . Let  $m(G)$  be the largest integer  $m$  such that  $G$  has at least  $m$  vertices of degree at least  $m - 1$ . A graph  $G$  is *tight* if it has exactly  $m(G)$  vertices of degree  $m(G) - 1$ .



Figure 1: The graph  $K_{3,3}^-$  with  $\varphi(K_{3,3}^-) = 2$  (on the left) and  $\varphi_r(K_{3,3}^-) = 3$  (on the right).

Another coloring parameter with domination constraints on the colors is the *b-chromatic number*. In a proper- $k$ -coloring, a vertex  $v$  of color  $i$  is a *b-vertex* if  $v$  is adjacent to at least one vertex colored  $j$ ,  $1 \leq j \neq i \leq k$ . A *b-k-coloring*, also called *b-coloring* when  $k$  is not specified, is a proper  $k$ -coloring such that every color class contains at least one b-vertex. The *b-chromatic number* of a graph  $G$ , denoted by  $\varphi(G)$ , is the largest integer  $k$  such that there exists a b- $k$ -coloring of  $G$ . In this paper, we introduce the concept of *b-relaxed number*, denoted by  $\varphi_r(G)$ . A *b-k-relaxed coloring* of  $G$  is a b- $k$ -coloring of a subgraph of  $G$ . The b-relaxed number of  $G$  is  $\varphi_r(G) = \max_{H \subseteq G} (\varphi(H))$ , for  $H$  an induced subgraph of  $G$ . Note that we have  $\varphi(G) \leq \varphi_r(G) \leq \partial\Gamma(G)$ . The difference between  $\varphi(G)$  and  $\varphi_r(G)$  can be arbitrary large. Let  $K_{n,n}^-$  denotes the complete bipartite graph  $K_{n,n}$  in which we remove  $n - 1$  pairwise non incident edges (or  $n - 1$  edges of a perfect matching in  $K_{n,n}$ ) [1]. For this graph we have  $\varphi(K_{n,n}^-) = 2$  and  $\varphi_r(K_{n,n}^-) = n$  as Figure 1 illustrates it (for  $n = 3$ ).

The concept of b-coloring has been introduced by Irving and Manlove [17], and a large number of papers was published (see e.g. [8, 20]). The b-chromatic number of regular graphs has been investigated in a serie of papers ([6, 18, 21, 23]). Determining the b-chromatic number of a tight graph is NP-hard even for a connected bipartite graph [19] and a tight chordal graph [13].

In this paper, we study the decision problems b-COL, b-r-COL and pG-COL with parameter  $t$  from Table 1.

	b-COL	b-r-COL	G-COL	pG-COL
Question	Does $\varphi(G) \geq t$ ?	Does $\varphi_r(G) \geq t$ ?	Does $\Gamma(G) \geq t$ ?	Does $\partial\Gamma(G) \geq t$ ?
Complexity class	undetermined	XP	XP [24]	XP

Table 1: The different decision problems with input a graph  $G$  and parameter  $t$  and their complexity class.

A decision problem is in FPT with parameter  $t$  if there exists an algorithm which resolves the problem in time  $O(f(t) n^c)$ , for an instance of size  $n$ , a computable function  $f$  and a constant  $c$ . A decision problem is in XP with parameter  $t$  if

there exists an algorithm which resolves the problem in time  $O(f(t) n^{g(t)})$ , for an instance of size  $n$  and two computable functions  $f$  and  $g$ .

The concept of  $t$ -atom was introduced independently by Gyárfás et al. [12] and by Zaker [24]. The family of  $t$ -atoms is finite and the presence of a  $t$ -atom can be determined in polynomial time for a fixed  $t$ . The following definition is slightly different from the definitions of Gyárfás et al. or Zaker, insisting more on the construction of every  $t$ -atom (some  $t$ -atoms can not be obtained with the initial construction of Zaker).

**Definition 1.1** ([24]). *The family of  $t$ -atoms is denoted by  $\mathcal{A}_t^\Gamma$ , for  $t \geq 1$ , and is defined by induction. The family  $\mathcal{A}_1^\Gamma$  only contains  $K_1$ . A graph  $G$  is in  $\mathcal{A}_{t+1}^\Gamma$  if there exists a graph  $G'$  in  $\mathcal{A}_t^\Gamma$  and an integer  $m$ ,  $m \leq |V(G')|$ , such that  $G$  is composed of  $G'$  and an independent set  $I_m$  of order  $m$ , adding edges between  $G'$  and  $I_m$  such that every vertex in  $G'$  is connected to at least one vertex in  $I_m$ .*

Moreover, in the following sections, we say that a graph  $G$  in a family of graphs  $\mathcal{F}$  is *minimal*, if no graphs of  $\mathcal{F}$  is a proper induced subgraph of  $G$ . For example, a minimal  $t$ -atom  $A$  is a  $t$ -atom for which there are no  $t$ -atoms which are induced in  $A$  other than itself.

**Theorem 1** ([12, 24]). *For a given graph  $G$ ,  $\Gamma(G) \geq t$  if and only if  $G$  contains an induced minimal  $t$ -atom.*

In this paper we prove equivalent theorems for  $b$ -relaxed number and partial Grundy number. In contrast with the minimal  $t$ -atoms, we can not define the minimal  $t$ -atoms for  $b$ -coloring as the smallest graphs such that  $G$  satisfies  $\varphi(G) = t$  (also called  $b$ -critical graphs).

The paper is organized as follows: Section 2 is devoted to the definition of  $t$ -atoms for the partial Grundy coloring. This concept allows us to prove that the partial Grundy coloring problem is in XP with parameter  $t$ . Section 3 is similar to Section 2 but for  $b$ -relaxed-coloring. Section 4 is devoted to the concept of  $b$ -critical vertices and edges. Section 5 is about  $b$ -perfect graphs. Finally, Section 6 deals with graphs for which the  $b$ -relaxed and the  $b$ -chromatic numbers are equal.

## 2 Partial-Grundy- $t$ -atoms: $t$ -atoms for partial Grundy coloring

We start this section with the definition of  $t$ -atoms for partial Grundy coloring.

**Definition 2.1.** *Let  $t$  be an integer with  $t \geq 2$ . The family  $\mathcal{A}^{\partial\Gamma}(t)$  of partial-Grundy- $t$ -atoms ( $pG$ - $t$ -atoms, for short) is obtained as follows: we begin by taking an independent set of  $t$  vertices, denoted by  $C(t)$ , called the center of the  $pG$ - $t$ -atom and associating a different integer of  $\{1, \dots, t\}$  to each vertex of  $C(t)$ . Let  $c_k$  be the vertex of  $C(t)$  with associated integer  $k$ ,  $1 \leq k \leq t$ . Let  $k$  be an integer with  $1 \leq k \leq t$ . A  $\partial$ - $k$ -dominating operation, on a graph  $G$  containing  $C(t)$ , consists in adding a set  $D$  of  $\ell$  vertices, with  $0 \leq \ell \leq t - k$  and*

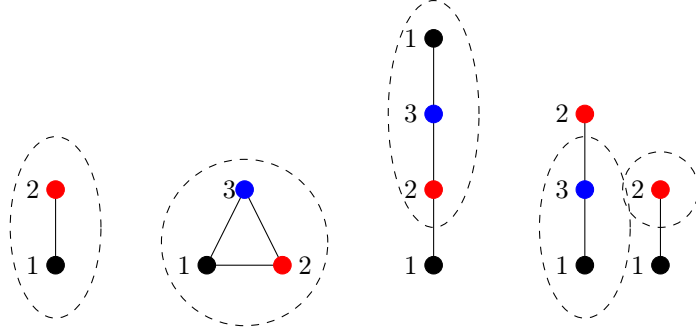


Figure 2: The minimal pG-2-atom (on the left) and the three minimal pG-3-atoms (the numbers are the colors of the vertices and the surrounded vertices form the centers).

associated integer  $k$  and adding edges between  $G - \{c_k\}$  and  $D \cup \{c_k\}$  such that every vertex of  $\{c_{k+1}, \dots, c_t\}$  is adjacent to a vertex of  $D \cup \{c_k\}$  and  $D \cup \{c_k\}$  remains independent. Let  $\mathcal{G}_1 = \{C(t)\}$ . The family of graphs  $\mathcal{G}_i$ , for  $2 \leq i \leq t$ , is obtained from the graphs in  $\mathcal{G}_{i-1}$  by doing a  $\partial$ - $i$ -dominating operation on these graphs. Finally, set  $\mathcal{A}^{\partial\Gamma}(t) = \mathcal{G}_t$ .

Note that the associated integers induce a partial Grundy coloring of the pG- $t$ -atom. These associated integers of the pG- $t$ -atoms are not considered anymore when we use the notion of induced subgraph. Figure 2 illustrates several pG- $t$ -atoms obtained using the previous construction.

**Observation 2.2.** For every graph  $G$  in  $\mathcal{A}^{\partial\Gamma}(t)$ , we have  $|V(G)| \leq \frac{t(t-1)}{2}$ .

**Lemma 2.3.** Let  $t$  and  $t'$  be two integers such that  $1 \leq t' < t$ . Every graph in  $\mathcal{A}^{\partial\Gamma}(t)$  contains a graph in  $\mathcal{A}^{\partial\Gamma}(t')$  as induced subgraph.

*Proof.* Every graph  $G$  in  $\mathcal{A}^{\partial\Gamma}(t)$  contains a graph  $G'$  of  $\mathcal{A}^{\partial\Gamma}(t')$ : we can obtain  $G'$  by removing every vertex with an associated integer  $k$  in  $G$ , for  $t' < k \leq t$ , and by removing, afterwards, the vertices of  $G'$  not adjacent to the centers.  $\square$

Note that the only minimal pG-2-atom is  $P_2$ . The minimal pG-3-atoms are  $C_3$ ,  $P_4$  and  $P_2 \cup P_3$ . These graphs are illustrated in Figure 2.

**Theorem 2.** Let  $G$  be a graph. For a graph  $G$ , we have  $\partial\Gamma(G) \geq t$  if and only if  $G$  contains an induced minimal pG- $t$ -atom.

*Proof.* Suppose that  $\partial\Gamma(G) = t'$  with  $t' \geq t$ . Thus, there exists a partial Grundy coloring of  $G$  with  $t'$  colors. Let  $u_1, \dots, u_{t'}$  be a set of Grundy vertices, each in a different color class of  $V(G)$ . The graph induced by  $N[u_1] \cup \dots \cup N[u_{t'}]$  contains a pG- $t'$ -atom. Hence, by Lemma 2.3, since  $G$  contains an induced pG- $t'$ -atom, then it also contains an induced minimal pG- $t$ -atom.

Suppose  $G$  contains an induced minimal pG- $t$ -atom. Thus, the integers associated to the vertices correspond to a partial-Grundy coloring of this pG- $t$ -atom. We can extend this coloring to a partial Grundy coloring of  $G$  with at least  $t$  colors, using a greedy algorithm [9] for the remaining vertices.  $\square$

**Proposition 2.4.** *Let  $G$  be a graph of order  $n$  and let  $t$  be an integer. There exists an algorithm in time  $O(n^{\frac{t(t-1)}{2}})$  to determine if  $\partial\Gamma(G) \geq t$ . Hence, the problem pG-COL with parameter  $t$  is in XP.*

*Proof.* By Theorem 2, it suffices to verify that  $G$  contains an induced minimal pG- $t$ -atom to have  $\partial\Gamma(G) \geq t$ . Since the size of a minimal pG- $t$ -atom is bounded by  $\frac{t(t-1)}{2}$ , we obtain an algorithm in time  $O(n^{\frac{t(t-1)}{2}})$ .  $\square$

We finish this section by determining every graph  $G$  with  $\partial\Gamma(G) = 2$ .

**Proposition 2.5.** *For a graph  $G$ , we have  $\partial\Gamma(G) = 2$  if and only if  $G = K_{n,m}$ , for some  $n \geq 1$  and  $m \geq 1$ .*

*Proof.* Zaker [24] has proven that  $\Gamma(G) = 2$  if and only if  $G$  is the disjoint union of copies of some  $K_{n,m}$ , for  $n \geq 1$  and  $m \geq 1$ . Let  $n$  and  $m$  be positive integers. We can note that a graph containing two disjoint copies of  $K_{n,m}$  contains an induced  $P_2 \cup P_3$ , hence a pG-3-atom. Hence, if  $\partial\Gamma(G) = 2$ , then  $G = K_{n,m}$ .

Moreover,  $K_{n,m}$  does not contain induced  $C_3$ ,  $P_4$  and  $P_3 \cup P_2$ . Hence,  $\partial\Gamma(K_{n,m}) = 2$ .  $\square$

### 3 b- $t$ -atoms: $t$ -atoms for b-coloring

As in the previous section, we start this section with the definition of b- $t$ -atoms (the notion of  $t$ -atom for b-coloring).

**Definition 3.1.** *Let  $t$  be an integer with  $t \geq 2$ . The family of b- $t$ -atoms  $\mathcal{A}^\varphi(t)$  is obtained as follows: we begin by taking an independent set of  $t$  vertices, denoted by  $C(t)$ , called the center of the b- $t$ -atom and associating a different integer of  $\{1, \dots, t\}$  to each vertex of  $C(t)$ . Let  $c_k$  be the vertex of  $C(t)$  with associated integer  $k$ ,  $1 \leq k \leq t$ . Let  $k$  be an integer with  $1 \leq k \leq t$ . A  $k$ -dominating operation on a graph  $G$  containing  $C(t)$ , consists in adding a set  $D$  of  $\ell$  vertices, with  $0 \leq \ell \leq t - 1$  and associated integer  $k$  and adding edges between  $G - \{c_k\}$  and  $D \cup \{c_k\}$  such that every vertex of  $C(t) - \{c_k\}$  is adjacent to a vertex of  $D \cup \{c_k\}$  and  $D \cup \{c_k\}$  remains independent. Let  $\mathcal{G}_1 = \{C(t)\}$ . The family of graphs  $\mathcal{G}_i$ , for  $2 \leq i \leq t$ , is obtained from the graphs in  $\mathcal{G}_{i-1}$  by doing an  $i$ -dominating operation on these graphs. Finally, set  $\mathcal{A}^\varphi(t) = \mathcal{G}_t$ .*

*For a fixed b- $t$ -atom, the set of possible associated integers is called the set of b- $t$ -atom colorings.*

Note that the associated integers induce a b-coloring of the b- $t$ -atom. These associated integers of the b- $t$ -atom are not considered anymore when we use

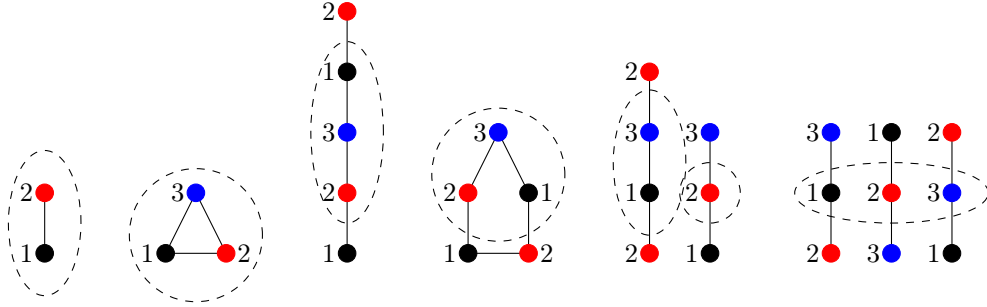


Figure 3: The minimal b-2-atom (on the left) and the five minimal b-3-atoms.

the notion of induced subgraph. Figure 3 illustrates several b- $t$ -atoms obtained using the previous construction.

**Observation 3.2.** For every graph  $G$  in  $\mathcal{A}^\varphi(t)$ , we have  $|V(G)| \leq t^2$ .

**Lemma 3.3.** Let  $t$  and  $t'$  be two integers such that  $1 \leq t' < t$ . Every graph in  $\mathcal{A}^\varphi(t)$  contains a graph in  $\mathcal{A}^\varphi(t')$  as induced subgraph.

*Proof.* Every graph  $G$  in  $\mathcal{A}^\varphi(t)$  contains a graph  $G'$  in  $\mathcal{A}^\varphi(t')$ : we can obtain  $G'$  by removing every vertex with an associated integer  $k$  in  $G$ , for  $t' < k \leq t$ , and by removing, afterwards, the vertices not adjacent to the centers.  $\square$

Note that the only minimal b-2-atom is  $P_2$ . The minimal b-3-atoms are  $C_3$ ,  $P_5$ ,  $C_5$ ,  $P_3 \cup P_4$  and  $P_3 \cup P_3 \cup P_3$ . These graphs are illustrated in Figure 3.

**Observation 3.4.** Every minimal pG- $t$ -atom is an induced subgraph of a minimal b- $t$ -atom or a minimal  $t$ -atom (an atom for the Grundy number).

**Proposition 3.5.** Let  $G$  be a graph. If  $\varphi(G) \geq t$ , then  $G$  contains an induced minimal b- $t$ -atom.

*Proof.* Suppose that  $\varphi(G) = t'$ , with  $t' \geq t$ . Thus, there exists a b-coloring of  $G$  with  $t'$  colors. Let  $u_1, \dots, u_{t'}$  be a set of b-vertices, each in a different color class of  $V(G)$ . The graph induced by  $N[u_1] \cup \dots \cup N[u_{t'}]$  contains a b- $t'$ -atom. Hence, by Lemma 3.3, since  $G$  contains an induced b- $t'$ -atom, then it also contains an induced minimal b- $t$ -atom.  $\square$

**Theorem 3.** For a graph  $G$ , we have  $\varphi_r(G) \geq t$  if and only if  $G$  contains an induced minimal b- $t$ -atom.

*Proof.* Suppose that the graph  $G$  contains an induced b- $t$ -atom  $A$ . Since  $A$  admits, by definition, a b- $t$ -coloring, we have  $\varphi_r(G) \geq t$ . Using Proposition 3.5, we obtain the reciprocal.  $\square$

**Definition 3.6.** Let  $G$  be a graph. For an induced subgraph  $A$  of  $G$ , let  $N(A) = \{v \in V(G) \setminus V(A) \mid uv \in E(G), u \in V(A)\}$ . A  $b$ - $t$ -atom  $A$  is feasible in  $G$  if there exists a  $b$ - $t$ -coloring of  $V(A)$  that can be extended to the vertices of  $N(A)$  without using new colors.

**Proposition 3.7.** Let  $G$  be a graph. If  $G$  contains an induced feasible minimal  $b$ - $t$ -atom and no induced feasible minimal  $b$ - $t'$ -atom, for  $t' > t$ , then  $\varphi(G) = t$ .

*Proof.* If  $G$  contains no induced minimal  $b$ - $t$ -atom, then by Proposition 3.5, we have  $\varphi(G) < t$ . Suppose that  $G$  contains an induced feasible minimal  $b$ - $t$ -atom  $A$  and no  $b$ - $t$ -coloring of  $G$  exists. We begin by considering that the vertices of  $A \cup N(A)$  are already colored with  $t$  colors. We can note that, by assumption, no coloring of  $A \cup N(A)$  (from the definition) can be extended to the whole graph using only  $t$  colors. Let  $t'$  be the largest integer such that the coloring can not be extended to a  $b$ - $t'$ -coloring of the whole graph and let  $v$  be a vertex that can not be given a color among  $\{1, \dots, t'\}$ . Thus, we suppose that the coloring can be extended to a  $b$ - $(t' + 1)$ -coloring where  $v$  is colored by  $t' + 1$ . Since  $A \cup N(A)$  is already colored, we have  $v \in V(G) \setminus (A \cup N(A))$ . The vertex  $v$  should be adjacent to vertices of every color, otherwise it could be colored. One vertex of each color class in  $N(v)$  should be adjacent to vertices of each color class (except its color). Otherwise, the colors of the vertices of  $N(v)$  could be changed in order  $v$  be colored. Let  $N_2[v]$  be the set of vertices at distance at most 2 from  $v$ . The set  $N_2[v]$  induces a graph containing a  $b$ - $t'$ -atom where  $N[v]$  contains the center of this  $b$ - $(t' + 1)$ -atom. Moreover, this  $b$ - $(t' + 1)$ -atom is feasible as the whole graph is  $b$ - $(t' + 1)$ -colorable, contradicting the hypothesis.  $\square$

**Proposition 3.8.** Let  $G$  be a graph. If  $\varphi(G) = t$ , then  $G$  contains an induced feasible minimal  $b$ - $t$ -atom and no induced feasible minimal  $b$ - $t'$ -atom, for  $t' > t$ .

*Proof.* Suppose  $\varphi(G) = t$ . By Proposition 3.5,  $G$  contains an induced minimal  $b$ - $t$ -atom. If no induced minimal  $b$ - $t$ -atom is feasible, then there exists no  $b$ - $t$ -coloring of  $G$ , a contradiction.  $\square$

A direct consequence of Proposition 3.7 and Proposition 3.8 is the following.

**Theorem 4.** For a graph  $G$ , we have  $\varphi(G) = t$  if and only if  $G$  contains an induced feasible minimal  $b$ - $t$ -atom and no induced feasible minimal  $b$ - $t'$ -atom, for  $t' > t$ .

Note that the problem of determining if a graph has a  $b$ - $t$ -coloring is NP-complete even if  $t$  is fixed [22]. However, it does not imply that determining if  $\varphi(G) \geq t$  for a graph  $G$  is NP-complete. In contrast with the  $b$ -chromatic number, determining if a graph has  $b$ -relaxed number at least  $t$  is in XP.

**Proposition 3.9.** Let  $G$  be a graph of order  $n$  and let  $t$  be an integer. There exists an algorithm in time  $O(n^{t^2})$  to determine if  $\varphi_r(G) \geq t$ . In particular, the problem  $b$ - $r$ -COL with parameter  $t$  is in XP.



*Proof.* It suffices to verify that  $G$  contains an induced minimal  $b$ - $t$ -atom to determine if  $\varphi_r(G) \geq t$ . By Observation 3.2, the size of a minimal  $b$ - $t$ -atom is bounded by  $t^2$ . Thus, we obtain an algorithm in time  $O(n^{t^2})$ .  $\square$

Another NP-complete problem for a graph  $G$  is to determine its  $b$ -spectrum [2], i.e. the set of integers  $k$  such that  $G$  is  $b$ - $k$ -colorable. For a graph  $G$  satisfying  $\varphi(G) = \varphi_r(G)$ , our algorithm can be used as the two concepts became the same.

**Corollary 3.10.** *Let  $G$  be a graph of order  $n$  and  $t$  be an integer. There exists an algorithm in time  $O(n^{t^2})$  to determine if  $\varphi(G) \geq t$  for a graph  $G$  satisfying  $\varphi(G) = \varphi_r(G)$ .*

## 4 $b$ -critical vertices and edges

The concept of  $b$ -critical vertices and  $b$ -critical edges has been introduced recently and since five years a large number of articles are considering this subject [1, 4, 5, 10, 25]. In this section, we illustrate how this notion is strongly connected with the concept of  $b$ - $t$ -atom.

**Definition 4.1** ([4, 10]). *Let  $G$  be a graph. A vertex  $v$  of  $G$  is  $b$ -critical if  $\varphi(G - v) < \varphi(G)$ . An edge  $e$  is  $b$ -critical if  $\varphi(G - e) < \varphi(G)$ . A vertex  $v$  (edge  $e$ , respectively) in a graph  $G$  is a  $b$ - $t$ -trap, if there exists a  $b$ - $t$ -atom of  $G$  that becomes feasible by removing  $v$  ( $e$ , respectively).*

**Proposition 4.2.** *Let  $G$  be a graph. A vertex  $v$  is  $b$ -critical if and only if it is in every feasible minimal  $b$ - $\varphi(G)$ -atom and  $v$  is not a  $b$ - $\varphi(G)$ -trap.*

*Proof.* Let  $t = \varphi(G)$ . First, if  $v$  is not in a feasible minimal  $b$ - $t$ -atom, then  $\varphi(G - v) = t$  and  $v$  is not  $b$ -critical. If  $v$  is a  $b$ - $t$ -trap, then, by definition,  $\varphi(G - v) = t$ . Second, suppose  $v$  is not a  $b$ - $t$ -trap. If  $v$  is in every feasible minimal  $b$ - $t$ -atom, then, since every minimal  $b$ - $t$ -atom in  $G$  does not contain any other feasible minimal  $b$ - $t$ -atom as induced subgraph,  $G - v$  does not contain a feasible minimal  $b$ - $t$ -atom. Thus,  $v$  is  $b$ -critical.  $\square$

**Corollary 4.3.** *If a graph  $G$  contains two induced feasible minimal  $b$ - $\varphi(G)$ -atoms with disjoint set of vertices, then it contains no  $b$ -critical vertex.*

**Corollary 4.4.** *Let  $G$  be a graph and  $v$  be a vertex of  $V(G)$ . If  $\varphi(G - v) > \varphi(G)$ , then  $G$  contains a minimal  $b$ - $t$ -atom which is not feasible, for some  $t > \varphi(G)$ . If  $\varphi(G - v) < \varphi(G) - 1$ , then  $G - v$  contains no feasible minimal  $b$ - $t$ -atom, for  $\varphi(G - v) < t \leq \varphi(G)$ .*

In [1], Balakrishnan and Raj have proved the following theorem.

**Theorem 5** ([1]). *Let  $G$  be a graph and  $v$  be a vertex of  $V(G)$ . We have  $\varphi(G) - \lfloor \frac{|V(G)|}{2} \rfloor + 2 \leq \varphi(G - v) \leq \varphi(G) + \lfloor \frac{|V(G)|}{2} \rfloor - 2$ .*

Moreover, some families of graphs which satisfy properties from Theorem 5 are determined. More precisely, they have determined the families of graphs for which there exists a vertex  $v$  such that  $\varphi(G - v) = \varphi(G) - \lfloor \frac{|V(G)|}{2} \rfloor + 2$  or  $\varphi(G - v) = \varphi(G) + \lfloor \frac{|V(G)|}{2} \rfloor - 2$ . In contrast with the b-chromatic number, we have the following property about the b-relaxed number.

**Proposition 4.5.** *Let  $G$  be a graph. If a vertex  $v$  is b-critical, then  $\varphi_r(G - v) = \varphi_r(G) - 1$ .*

*Proof.* By Proposition 4.2,  $v$  is in every b- $\varphi(G)$ -atom. Let  $i$  be the integer associated to  $v$  in the construction of this b- $\varphi(G)$ -atom. By removing the vertices with associated integer  $i$ , we obtain a b- $(\varphi(G) - 1)$ -atom and thus  $\varphi_r(G - v) = \varphi_r(G) - 1$ .  $\square$

Note that this proposition was already proved for trees [4] (a tree  $T$  satisfies  $\varphi(T) = \varphi_r(T)$ , since every b-coloring of a subtree can be extended to the whole tree).

**Lemma 4.6.** *Let  $G$  be a graph with  $4 \leq |V(G)| \leq 5$  and  $E(G) \neq \emptyset$ . We have  $\varphi_r(G - v) = \varphi_r(G) + \lfloor \frac{|V(G)|}{2} \rfloor - 2$ , for every vertex  $v$  of  $V(G)$ , if and only if  $G$  contains two disjoint edges but no induced minimal b-3-atom.*

*Proof.* We can note that we have  $\varphi_r(G - v) = \varphi_r(G) + \lfloor \frac{|V(G)|}{2} \rfloor - 2$  if and only if  $\varphi_r(G - v) = \varphi_r(G)$ .

First, if  $G$  contains no minimal b-3-atom and contains an edge, then  $\varphi_r(G) = 2$ . Moreover, if  $G$  contains two disjoint edges, then for any vertex  $v$ ,  $G - v$  contains  $P_2$  and  $\varphi_r(G - v) = 2$ .

Second, suppose that for every vertex  $v$ ,  $\varphi_r(G - v) = \varphi_r(G)$ . The only minimal b-3-atoms that contains at most five vertices are  $K_3$ ,  $C_5$  and  $P_5$ . Moreover, the only minimal b-4-atoms and b-5-atoms that contain at most five vertices are  $K_4$  and  $K_5$ . We are going to show that  $G$  is not one of these graphs

**Case 1:**  $\varphi_r(G) = 5$ . If  $G$  is a  $K_5$ , then, by removing any vertex  $v$ , we obtain  $\varphi_r(G - v) = 4$ .

**Case 2:**  $\varphi_r(G) = 4$ . If  $G$  is a  $K_4$ , then, by removing any vertex  $v$ , we obtain  $\varphi_r(G - v) = 3$ . If  $G$  contains an induced  $K_4$ ,  $|V(G)| = 5$  and  $G$  is not  $K_5$ , then there exists a vertex  $v$  such  $G - v$  has no induced  $K_4$  and  $\varphi_r(G - v) = 3$ .

**Case 3:**  $\varphi_r(G) = 3$ . If  $G$  contains an induced  $K_3$  and no induced  $K_4$ , then, since the induced  $K_3$  in  $G$  have a common vertex  $v$ , we obtain  $\varphi_r(G - v) = 2$ . Moreover, if  $G$  is  $P_5$  or  $C_5$ , then, by removing any vertex  $v$ , we obtain  $\varphi_r(G - v) = 2$ .

Thus, we can suppose that  $\varphi_r(G) = 2$ . If  $G$  contains only edges with a common vertex  $v$ , then  $\varphi_r(G - v) = 1$ . Hence,  $G$  contains no b-3-atom and contain two disjoint edges.  $\square$

The following theorem is a generalization of a conjecture of Blidia et al. [3] for the parameter  $\varphi_r$ . Note that the graphs  $P_4$ ,  $C_4$  and  $P_2 \cup P_2$  do not contain any induced minimal b-3-atom and contain two disjoint edges.

**Theorem 6.** *Let  $G$  be a graph. We have  $\varphi_r(G - v) = \varphi_r(G) + \lfloor \frac{|V(G)|}{2} \rfloor - 2$ , for every vertex  $v$  of  $V(G)$ , if and only one of these conditions is true about  $G$ :*

- i)  $G$  is  $P_2$  or  $C_3$ .
- ii)  $E(G) = \emptyset$  and  $4 \leq |V(G)| \leq 5$ .
- iii)  $4 \leq |V(G)| \leq 5$  and  $G$  contains two disjoint edges but no b-3-atom.

*Proof.* Note that if  $|V(G)| \geq 6$ , then, by Proposition 4.5, we can not have  $\varphi_r(G - v) < \varphi_r(G) + \lfloor |V(G)|/2 \rfloor - 2$ . Note also that if  $G$  contains only one vertex, then it can not satisfy  $\varphi_r(G - v) < \varphi_r(G) + \lfloor |V(G)|/2 \rfloor - 2$ .

First, if  $2 \leq |V(G)| \leq 3$ , then we have  $\varphi_r(G - v) = \varphi_r(G) - 1$  if and only if  $G$  is a minimal b- $t$ -atom. Hence, if and only if  $G$  is  $P_2$  or  $C_3$ . Second, if  $G$  contains no edges, then  $\varphi_r(G) = 1$  and for any vertex  $v$ ,  $\varphi_r(G - v) = 1$ . The third condition is obtained by Lemma 4.6.  $\square$

**Definition 4.7.** *Let  $t$  be a positive integer and  $A$  be a b- $t$ -atom. An edge  $e$  is b-atom-critical in  $A$  if  $A - e$  is not a b- $t$ -atom.*

**Proposition 4.8.** *Let  $G$  be a graph. An edge  $e$  is b-critical if and only if it is b-atom-critical in every feasible minimal b- $\varphi(G)$ -atom and  $e$  is not a b- $\varphi(G)$ -trap.*

*Proof.* Let  $t = \varphi(G)$ . First, if  $e$  is not b-atom-critical in a feasible minimal b- $t$ -atom, then  $G - e$  contains a feasible minimal b- $t$ -atom and  $\varphi(G - e) = t$ . If  $e$  is a b- $t$ -trap, then, by definition,  $\varphi(G - e) = t$ . Second, suppose that  $e$  is not a b- $t$ -trap. If  $e$  is b-atom-critical in every feasible minimal b- $t$ -atom, then, since every feasible minimal b- $t$ -atom in  $G$  does not contain any other feasible minimal b- $t$ -atom as subgraph in  $G - e$ , the graph  $G - e$  does not contain a feasible minimal b- $t$ -atom. Thus,  $e$  is b-critical.  $\square$

**Corollary 4.9.** *If a graph  $G$  contains two induced feasible minimal b- $\varphi(G)$ -atoms with disjoint sets of b-atom-critical edges, then  $G$  contains no b-critical edge.*

## 5 b-perfect graphs

A b-perfect graph is a graph for which every induced subgraph satisfies that its b-chromatic number is equal to its chromatic number. More generally, we present the following definitions.

**Definition 5.1** ([14]). *A graph  $G$  is b- $\chi$ - $k$ -bounded, for  $k$  a positive integer, if  $\varphi(G') - \chi(G') \leq k$ , for every induced subgraph  $G'$  of  $G$ . A b- $\chi$ -0-bounded graph is called a b-perfect graph. A  $\chi$ - $k$ -unbounded b-atom, for  $k$  a positive integer,*

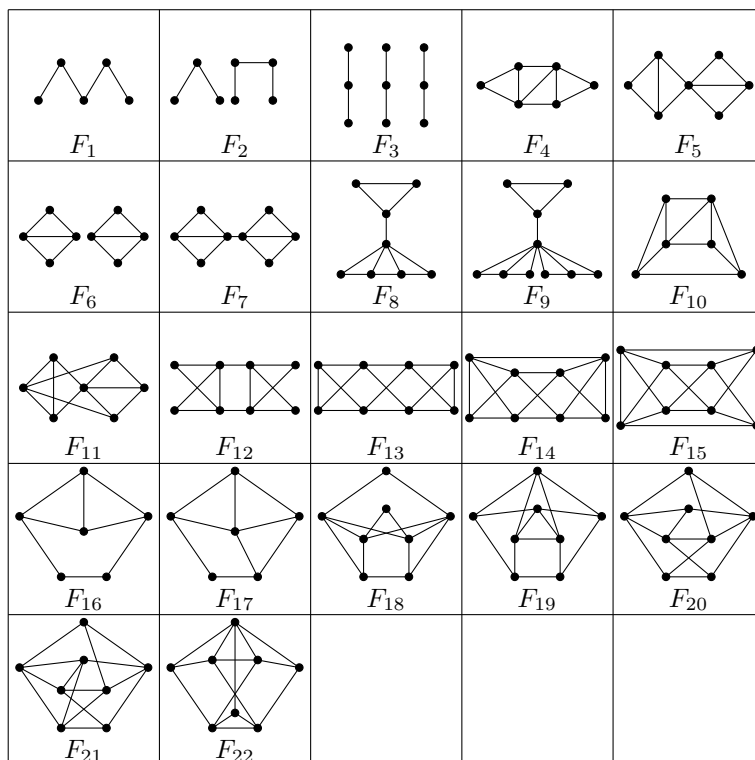


Figure 4: The family  $\mathcal{F}$ : the imperfect b-atoms [15].

is a b-atom which is not  $b$ - $\chi$ - $k$ -bounded and which does not contain any  $b$ - $\chi$ - $k$ -bounded induced minimal b-atom other than itself. An imperfect b-atom is a b-atom which is not b-perfect and which does not contain any induced minimal b-atom which is not b-perfect other than itself.

Hoang et al. [15] characterized b-perfect graphs by giving the family  $\mathcal{F}$  of forbidden induced subgraphs depicted in Figure 4. We recall the following theorem:

**Theorem 7** ([15]). *A graph is b-perfect if and only if it contains no graph from  $\mathcal{F}$  as induced subgraph.*

Note that every graph in the family  $\mathcal{F}$  is a b- $t$ -atom for some  $t$ . More precisely,  $F_1$ ,  $F_2$  and  $F_3$  are the only minimal bipartite b-3-atoms. The remaining graphs are minimal b-4-atoms that do not contain  $F_1$ ,  $F_2$  and  $F_3$  as induced subgraph and which admit a proper coloring with three colors (as mentioned in [16]). We can state the following property about b- $t$ -atoms.

**Observation 5.2.** *Let  $t$  and  $t'$  be two integers with  $t' < t$ . If a minimal b- $t$ -atom  $A$  contains an induced  $\chi$ - $k$ -unbounded b- $t'$ -atom, then  $A$  is not  $b$ - $\chi$ - $k$ -bounded.*

**Theorem 8.** *Let  $k$  be a positive integer. A graph  $G$  is not  $b$ - $\chi$ - $k$ -bounded if and only if it contains a  $\chi$ - $k$ -unbounded  $b$ -atom.*

*Proof.* First, if  $G$  contains a  $\chi$ - $k$ -unbounded  $b$ -atom, then, by definition,  $G$  is not  $\chi$ - $k$ -bounded.

Second, suppose  $G$  is not  $b$ - $\chi$ - $k$ -bounded. Then, there exists an induced subgraph  $A$  of  $G$  of minimal order which is not  $b$ - $\chi$ - $k$ -bounded. By removing vertices of  $A$  we can only decrease the chromatic number. Thus, by removing vertices we can obtain a  $b$ - $\varphi(A)$ -atom which is  $\chi$ - $k$ -unbounded.  $\square$

**Corollary 5.3.** *The graphs with  $b$ -chromatic number  $t$  which are  $b$ - $\chi$ - $k$ -bounded, for fixed integers  $k$  and  $t$ , can be defined by forbidding a finite family of induced subgraphs: the  $\chi$ - $k$ -unbounded  $b$ -atoms. Hence, a graph  $G$  is  $b$ -perfect if and only if it does not contain imperfect  $b$ -atoms.*

Let  $b$ - $\chi$ -BOUNDED be the following decision problem with parameter  $\varphi(G)$  and let  $k$  be an integer, with  $0 \leq k < \varphi(G)$ .

**$b$ - $\chi$ - $k$ -BOUNDED**

**Instance :** A graph  $G$ .

**Parameter :**  $\varphi(G)$ .

**Question:** Does  $\varphi(G) - \chi(G) \geq k$ ?

By Corollary 5.3, we obtain the following corollary:

**Corollary 5.4.** *Let  $G$  be a graph and  $k$  be an integer, with  $0 \leq k < \varphi(G)$ . There exists an algorithm in time  $O(n^{\varphi(G)^2})$  to solve  $b$ - $\chi$ - $k$ -BOUNDED.*

Since a graph  $G$  is  $b$ -perfect if and only if it does not contain imperfect  $b$ -atoms, we have the following theorem:

**Theorem 9.** *The number of imperfect  $b$ -atoms is finite. A graph is an imperfect  $b$ -atom if and only if it is in the family  $\mathcal{F}$  (Figure 4).*

The previous theorem is a consequence of Theorem 7. By proving that every  $b$ -4-atom except  $K_4$  contains an induced subgraph of the family  $\mathcal{F}$  and by using Proposition 5.2, we can possibly obtain another proof of Theorem 7.

## 6 $b$ -chromatic and $b$ -relaxed chromatic numbers

In this section we consider the  $b$ -relaxed number relatively to the  $b$ -chromatic number. About the relation between these parameters we can state the following properties.

**Proposition 6.1.** *For a graph  $G$  satisfying  $\varphi(G) = \Delta(G) + 1$ , we have  $\varphi(G) = \varphi_r(G)$ .*

*Proof.* Since every vertex is adjacent to at most  $\Delta(G)$  neighbors, it is always possible to extend a  $(\Delta + 1)$ -coloring to a new vertex.  $\square$

**Proposition 6.2.** For a tree  $T$ , we have  $\varphi(T) = \varphi_r(T)$ .

*Proof.* It is always possible to extend a  $k$ -coloring to a new vertex in a tree.  $\square$

We finish this paper by proving that when a graph  $G$  has sufficiently large girth, we have  $\varphi(G) = \varphi_r(G)$ .

**Theorem 10.** Let  $G$  be a graph with girth  $g$  and  $\varphi_r(G) \geq 3$ . If  $g \geq 7$ , then  $\varphi(G) = \varphi_r(G)$ .

*Proof.* Suppose that  $\varphi_r(G) = t$ . Let  $A_t$  be a colored  $b$ - $t$ -atom of  $G$  and let  $c_1, \dots, c_t$  be the vertices of the center of this  $b$ - $t$ -atom. Moreover, we suppose that  $c_i$  is colored by  $i$ . If  $A_t$  is not feasible, then there exists a subset of vertices in  $N(A_t)$  such that this coloring can not be extended to these vertices. By assumption, we suppose that  $A_t$  is not feasible. For a vertex  $u \in N(A_t)$ , we denote by  $N_c(u)$  the set  $\{i \in \{1, \dots, t\} \mid \exists v \in N(u) \cap N[c_i]\}$ . We denote by  $N[A_t]$ , the set of vertices  $A_t \cup N(A_t)$ . In order to have a contradiction we begin by proving the following properties:

- i) if  $u \in N(A_t)$ , then  $u$  is not adjacent to two vertices of  $N[c_i]$ , for  $1 \leq i \leq t$ ;
  - ii) if  $u, v \in N(A_t)$  and  $N_c(u) \cap N_c(v) \neq \emptyset$ , then  $u$  and  $v$  are at distance at least 3;
  - iii) if  $u, v \in N[c_i]$  and  $u', v' \in N[c_i]$  for some  $i$ ,  $1 \leq i \leq t$ , then no two pairs of vertices among  $\{u, v, u', v'\}$  are adjacent.
- i) If  $u$  is adjacent to two vertices of  $N[c_i]$ , for some  $i$ ,  $1 \leq i \leq t$ , then  $u$  is in a cycle of length at most 4.
  - ii) If two vertices  $u$  and  $v$  from  $N(A_t)$  are at distance at most 2 and are both adjacent to vertices of  $N[c_i]$ , for  $1 \leq i \leq t$ , then  $u$  and  $v$  are in a cycle of length at most 6.
  - iii) If two pairs of vertices among  $\{u, v, u', v'\}$  are adjacent then there is a cycle of length at most 6 in  $G$ .

We are going to prove that either each vertex  $u \in N(A_t)$  can be given a color from  $\{1, \dots, t\}$  or the graph contains a  $b$ - $(t+1)$ -atom (which contradicts  $\varphi_r(G) = t$ ). First, we will consider the vertices from  $N(A_t)$  which are not only adjacent to centers. Afterward, we will consider the remaining vertices of  $N(A_t)$ .

**Case 1:**  $u$  has at least one neighbor in  $A_t$  which is not a center.

By properties i) and ii),  $u$  has at most  $t$  neighbors in  $N[A_t]$ . If  $u$  has less than  $t$  neighbors in  $N[A_t]$ , then it can be given a color, hence assume  $u$  has exactly  $t$  neighbors in  $N[A_t]$ . If  $u$  has less than  $t$  neighbors in  $A_t$ , then there exists a neighbor  $w$  of  $u$  in  $N(A_t)$ . By property ii),  $|N_c(w)| = 1$ , i.e.  $w$  has two neighbors in  $N[A_t]$  and thus can be recolored in order to give its original color to  $u$ . If  $u$  has exactly  $t$  neighbors in  $A_t$ , let  $v_1, \dots, v_\ell$  be

the neighbors of  $u$  in  $A_t$  which are not in the center. Let  $c'$  be the color of  $v_1$  and let  $c''$  be the color of  $v_2$  if it exists. Moreover, let  $i_1$  and  $i_2$  be two integers such that  $v_1 \in N(c_{i_1})$  and  $v_2 \in N(c_{i_2})$  (if  $\ell \geq 2$ ).

**Subcase 1.a:** suppose that  $\ell = 1$ , i.e.,  $u$  is adjacent to  $t - 1$  vertices from the center and to one vertex in  $A_t$  which is not a center.

We can note that no vertex from  $N[c_{i_1}]$  can have a neighbor in  $A_t$ , since it would create a cycle of length at most 6. Consequently, we can exchange the color of  $v_1$  with the color of one vertex from  $N(c_{i_1})$  and color  $u$  by  $c'$ .

**Subcase 1.b:** suppose that  $\ell \geq 2$ , i.e.,  $u$  is adjacent to at most  $t - 2$  vertices from the center and to at least two vertices in  $A_t$  which are not centers.

The vertex  $v_1$  has no neighbor in  $A_t - N[c_{i_1}]$  and  $v_2$  has no neighbor in  $A_t - N[c_{i_2}]$  since it would create a cycle of length at most 6. If a vertex of  $N(c_{i_1}) - \{v_1\}$  has no neighbor of color  $c'$ , then we exchange the color of  $v_1$  with the color of this vertex and color  $u$  by  $c'$ . If a vertex of  $N(c_{i_2}) - \{v_2\}$  does not have a neighbor of color  $c''$ , then we exchange the color of  $v_2$  with the color of this vertex and color  $u$  by  $c''$ . Thus, we suppose that every vertex of  $N(c_{i_1}) - \{v_1\}$  has a neighbor of color  $c'$  and every vertex of  $N(c_{i_2}) - \{v_2\}$  has a neighbor of color  $c''$ .

In the case  $c'' = i_1$  and  $c' = i_2$ , by property iii), no vertex of  $N[c_{i_1}] \cup N[c_{i_2}]$  has a neighbor in  $A_t - N[c_{i_1}] \cup N[c_{i_2}]$  other than these vertices of color  $c'$  and  $c''$ . However, there could exist two adjacent vertices  $w$  and  $w'$  with  $w \in N[c_{i_1}]$  and  $w' \in N[c_{i_2}]$ . If  $t = 3$ , then  $w$  and  $w'$  are colored by the same color and thus can not be adjacent. Consequently, we can exchange the color of  $v_1$  with the color of  $v_2$ , the color of  $c_{i_1}$  with the color of  $c_{i_2}$  and afterward we can exchange the color of one vertex from  $N(c_{i_1}) - \{w, v_1\}$  (if  $t = 3$ , then we take  $w$ ) with the color of  $v_1$  and color  $u$  by  $c''$ .

In the case  $c'' \neq i_1$  (or  $c' \neq i_2$ , respectively), we suppose, without loss of generality, that  $c' = i_2$  (or  $c'' = i_1$ , respectively). In this case, we exchange the color of  $c_{i_1}$  with the color of  $c_{i_2}$  and the color of  $v_1$  with the color of a vertex  $w$  of color  $i_1$  in  $N(c_{i_2})$  (the color of  $v_2$  with the color of a vertex  $w$  of color  $i_2$  in  $N(c_{i_1})$ , respectively) and color  $u$  by  $c'$  (by  $c''$ , respectively). We can note that the vertex  $w$  in  $N(c_{i_2})$  (the vertex  $w$  in  $N(c_{i_1})$ , respectively) can be adjacent to a vertex of color  $c'$  (of color  $c''$ , respectively). However, if  $w$  is adjacent to a vertex of color  $c'$  (of color  $c''$ , respectively), then no vertex of  $N(c_{i_2}) - \{w, v_2\}$  (of  $N(c_{i_1}) - \{w, v_1\}$ , respectively) is adjacent to a vertex of color  $c'$  (of color  $c''$ , respectively). If  $w$  is adjacent to a vertex of this color, then we exchange the color of  $w$  with the color of a vertex of  $N(c_{i_2}) - \{w, v_2\}$  (of  $N(c_{i_1}) - \{w, v_1\}$ , respectively). We can remark that if  $t = 3$ , then  $w$  can not be adjacent to a vertex of color  $c'$  (of color  $c''$ , respectively) and the previous case never happen.

**Case 2:** every neighbor of  $u$  is in the center.

If  $u$  does not have neighbors in each color class, then we can give a color to  $u$ . The vertices from  $N(u) \cap A_t$  are centers, and thus these vertices

have neighbors from each color class. If every vertex  $v$  in  $N(u) \cap N(A_t)$  colored by a given color  $c$  does not have neighbors from each color class, then we recolor these vertices and give the color  $c$  to  $u$ . If  $u$  remains uncolorable, the set  $N_2[u]$  induces a graph containing a  $b-(t+1)$ -atom, which contradicts  $\varphi_r(G) = t$ .

□

We think that the previous theorem can be used to determine the family of graphs of girth at least 7 satisfying  $\varphi(G) = m(G)$ . It has already been proven that graphs of girth at least 7 have  $b$ -chromatic number at least  $m(G) - 1$  [7].

**Corollary 6.3.** *Let  $G$  be a graph of girth at least 7 and of order  $n$  and let  $t$  be an integer. There exists an algorithm in time  $O(n^{t^2})$  to determine if  $\varphi(G) \geq t$ .*

## 7 Open questions

We conclude this article by listing few open questions.

1. For which family of graphs are the  $b$ -relaxed number and the  $b$ -chromatic number equal?
2. Does there exist an easy characterization of feasible  $b$ - $t$ -atoms?
3. Does there exist an FPT algorithm, with parameter  $t$ , to determine if  $\varphi(G) \geq t$ ?

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