On S-packing edge-colorings of cubic graphs

Nicolas Gastineau, Olivier Togni

On S-packing edge-colorings of cubic graphs

Nicolas Gastineau[1,2] and Olivier Togni[1]

[1]LE2I FRE2005, CNRS, Arts et Métiers, Université Bourgogne
Franche-Comté, F-21000 Dijon, France
[2]PSL, Université Paris-Dauphine, LAMSADE UMR CNRS 7243, France

November 28, 2017

Abstract

Given a non-decreasing sequence $S = (s_1, s_2, \ldots, s_k)$ of positive integers, an S-*packing edge-coloring* of a graph G is a partition of the edge set of G into k subsets $\{X_1, X_2, \ldots, X_k\}$ such that for each $1 \leq i \leq k$, the distance between two distinct edges $e, e' \in X_i$ is at least $s_i + 1$. This paper studies S-packing edge-colorings of cubic graphs. Among other results, we prove that cubic graphs having a 2-factor are $(1, 1, 1, 3, 3)$-packing edge-colorable, $(1, 1, 1, 4, 4, 4, 4, 4)$-packing edge-colorable and $(1, 1, 2, 2, 2, 2, 2)$-packing edge-colorable. We determine sharper results for cubic graphs of bounded oddness and 3-edge-colorable cubic graphs and we propose many open problems.

1   Introduction

All the graphs considered in this paper are simple and connected, unless stated otherwise. A proper *edge-coloring* of a graph G is a mapping which associates a color (an integer) to each edge such that adjacent edges get distinct colors. In such a coloring, each color class is a matching (also called stable set of edges or 1-packing). According to Vizing's famous theorem, every cubic graph needs either 3 or 4 colors for a proper edge-coloring. The bridgeless cubic graphs (often with other restrictions) which are not edge-colorable with three colors are called *snarks* [5, 20].

As an extension, a *d-strong edge-coloring* of G is a proper coloring such that edges at distance at most d have distinct colors, i.e., a partition of $E(G)$ into sets of edges at pairwise distance at least $d + 1$, also called *d-packings*. A 2-strong edge-coloring is simply called a strong edge-coloring and a 2-packing of edges is an induced matching. Strong edge-colorings of cubic graphs retain a lot of attention since decades [1, 9, 19].

The aim of this paper is to study a mixing of these two types of edge-colorings, i.e., colorings of (sub)cubic graphs in which some color classes are 1-packings while other are
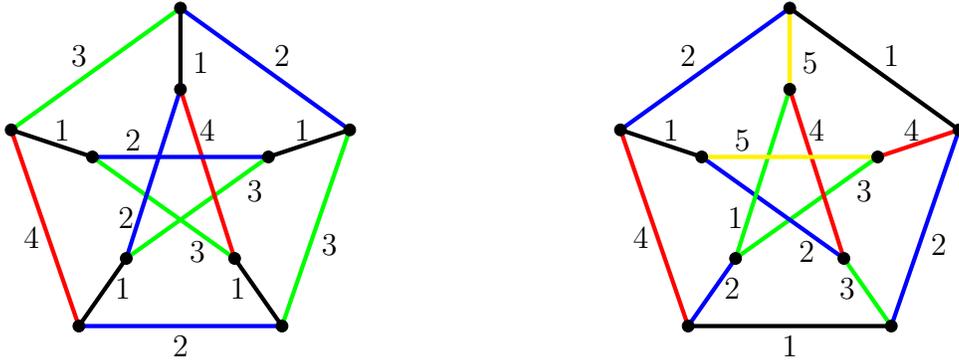
Figure 1: A $(1,1,1,2)$-coloring (on the left, with colors 1, 2 and 3 of radius 1 and color 4 of radius 2) and a $(1,1,2,2,2)$-coloring (on the right, with colors 1 and 2 of radius 1 and colors 4 and 5 of radius 2) of the Petersen graph.

d-packings, $d \geq 2$. More formally, given a non-decreasing sequence $S = (s_1, s_2, \ldots, s_k)$ of positive integers, an *S-packing edge-coloring* of a graph G is a partition of the edge set of G into k subsets $\{X_1, X_2, \ldots, X_k\}$ such that each X_i is an s_i-packing, $1 \leq i \leq k$.

The vertex analogous of S-packing edge-coloring has been first studied by Goddard and Xu [16, 17] and then recently on cubic graphs [2, 3, 4, 12, 14]. The particular case of $(1, 2, \ldots, k)$-packing coloring has been the subject of many papers (see [7, 8, 13]) since its introduction by Goddard et al. [18].

For an edge-coloring, a color for which the color class is an r-packing is said to be a color of *radius r*. In order to avoid long subsequences of the same integer in sequences of colors, we sometimes use the exponent to denote repetitions of an integer, e.g., $(1^2, 2^5) = (1, 1, 2, 2, 2, 2, 2)$. Also, to simplify, an S-packing edge-coloring will be simply called an *S-coloring* in the remainder of the paper. A $(1, 1, 1, 2)$-coloring and a $(1, 1, 2, 2, 2)$-coloring of the Petersen graph are illustrated in Figure 2 (one can check that the Petersen graph is not $(1, 1, 2, 2)$-colorable).

Let G be a graph and $A \subseteq E(G)$. By $G^k[A]$, we denote the graph with vertex set A and edge set $\{ee' \in E(G) \mid e \in A,\ e' \in A,\ d_G(e, e') \leq k\}$, where $d_G(e, e')$ is the usual distance between the two edges e and e' in G. We recall that a 2-*factor* of G is a spanning subgraph of G that consists in a disjoint union of cycles. For a cubic graph G having a 2-factor, the *oddness* of G is the minimum number of odd cycle among all 2-factors of G. According to Petersen's theorem, every bridgeless cubic graph has a 2-factor.

Definition 1.1. *For a graph G with a 2-factor $\mathscr{F}$ and a set $A \subseteq E(\mathscr{F})$, we use the following notation:*

    i) A is of type I if it contains exactly one edge per odd cycle of $\mathscr{F}$ and no edge of any even cycle of $\mathscr{F}$;

    ii) A is of type II if no two edges of A are adjacent in G and if A contains $\lfloor n/2 \rfloor$ edges in every cycle of length n from $\mathscr{F}$, $n \geq 3$.

These two definitions will be used several times in the paper in order to describe the edges that remain to be colored in a cubic graph in which a maximum number of edges are colored with one, two or three colors of radius 1. Notice that a set of type I is also called an odd cycle (edge) transversal of $\mathscr{F}$.

Table 1: The minimum integer $n = \ell + m$ in order that all cubic graphs (and all 3-edge-colorable cubic graphs) having a 2-factor are $(1^\ell, k^m)$-colorable (the bold numbers represent the exact values of n and a pair of two integers a-b represents a lower bound and an upper bound on n).

Class	cubic graphs			3-edge-colorable cubic graphs		
$k \backslash \ell$	1	2	3	1	2	3
2	8-10	6-7	**4** [11]	6-9	5-6	**3**
3	15-21	9-13	**5**	15-19	9-11	**3**
4	31-48	17-28	5-8	31-43	17-23	**3**

As any subcubic graph H is the subgraph of a cubic graph G and as $d_H(e, e') \geq d_G(e, e')$ for any two edges $e, e' \in E(H)$, then any S-coloring of G is also an S-coloring of H. Therefore, the results of this paper that are not concerned with oddness can be easily extended to subcubic graphs.

Table 1 summarizes the main results proven in this paper. Note that the lower bounds, except for the sequence $(1, 1, 1, 2)$ and the sequences $(1, 1, 1, \ldots)$ for 3-edge-colorable graphs, have been determined by computer.

The paper is organized as follows. In section 2, we begin by presenting structural results about sets of type I and the relation between S-coloring and sets of type I and II. We prove in Section 3 that cubic graphs having a 2-factor are $(1, 1, 1, 3, 3)$-colorable and conjecture that all cubic graphs are $(1, 1, 1, 3)$-colorable, except the Petersen and Tietze graphs. This conjecture is proven for some restricted classes of snarks. Similar results are given for sequences of type $(1, 1, 1, 4, \ldots, 4)$. In Section 4 we study $(1, 1, k, \ldots, k)$-colorings and prove that all cubics graphs having a 2-factor are $(1^2, 2^5)$-colorable and also colorable with two colors of radius one and a finite number of colors of radius k, for any $k \geq 2$. In Section 5 we prove that for a fixed integer k, every cubic graph having a 2-factor is $(1, k, \ldots, k)$-colorable with a finite number of occurrences of k in the sequence. Finally, in Section 6 we prove that for every positive integer k, there exists a subcubic graph which is not $(1, 2, \ldots, k)$-colorable.

2   Sets of type I and II

The following result from Fouquet and Vanherpe is a structural property about non 3-edge-colorable cubic graphs [11]. It will be used several times in the proofs of Section
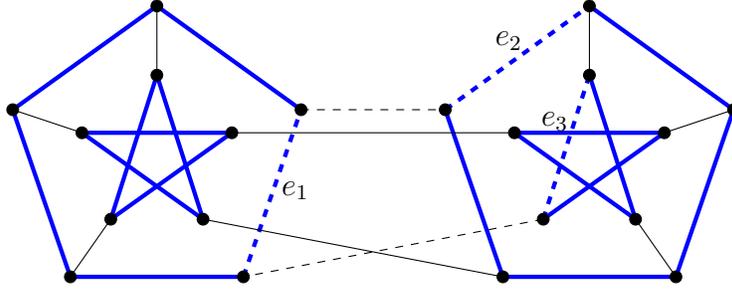
Figure 2: A 2-factor containing three edges e_1, e_2 and e_3 being in different odd cycles which induce a subgraph containing a path of length 5 (dashed lines: edges in the path of length 5, thick lines: edges from the 2-factor).

3.

Proposition 2.1 ([11]). *Let G be a cubic graph having a 2-factor. Let $\mathscr{F}$ be a 2-factor of G containing a minimum number of odd cycles and let $\mathscr{F}'$ be the set of odd cycles from $\mathscr{F}$. Then, no three edges being in different cycles of $\mathscr{F}'$ induce in G a subgraph containing a path of length 5.*

Figure 2 illustrates three edges in the forbidden configuration described in Proposition 2.1 in a fixed 2-factor of a cubic graph (by Proposition 2.1, it means that there exists a 2-factor containing at most two odd cycles in the graph from Figure 2). Note that the previous proposition was not explicitly presented in the paper of Fouquet and Vanherpe but can be easily obtained by combining Properties 3 and 6 of [11, Theorem 4]. Note also that the proof of Theorem 4 is written in another paper [10] from the same authors.

Lemma 2.2 ([11]). *Let G be a cubic graph having a 2-factor. Let $\mathscr{F}$ be a 2-factor of G containing a minimum number of odd cycles and let $\mathscr{F}'$ be the set of odd cycles from $\mathscr{F}$. There exists a set A of type I in $\mathscr{F}'$ such that $G^2[A]$ is an empty graph, i.e., $\chi(G^2[A]) \leq 1$.*

As the previous proposition, Lemma 2.2 has not been explicitly presented in the paper of Fouquet and Vanherpe but is an intermediate step to prove a Theorem [11, Theorem 8].

The following lemma will allow in Sections 3, 4 and 5, to reduce the problem of finding an S-coloring to the one of finding a set A of type I and/or II such that $G^k[A]$ has small chromatic number (where $2 \leq k \leq 4$ and k appears in S).

Lemma 2.3. *Let G be a cubic graph having a 2-factor $\mathscr{F}$ and let ℓ, ℓ' and ℓ'' be positive integers. Let $A \subseteq E(\mathscr{F})$, $B \subseteq E(\mathscr{F})$ and $C \subseteq E(\mathscr{F})$ be sets such that A, B and C form a partition of $E(\mathscr{F})$, A being of type I, B and C being of type II. The following properties hold:*

   i) if $\chi(G^k[A]) \leq \ell$, then G is $(1,1,1,k^\ell)$-colorable;

   ii) if $\chi(G^k[A]) \leq \ell$ and $\chi(G^k[B]) \leq \ell'$, then G is $(1,1,k^{\ell+\ell'})$-colorable;

4

iii) if $\chi(G^k[A]) \leq \ell$, $\chi(G^k[B]) \leq \ell'$ and $\chi(G^k[C]) \leq \ell''$, then G is $(1, k^{\ell+\ell'+\ell''})$-colorable.

Proof. Note that $G - \mathscr{F}$ is a matching and thus can be colored with a color of radius 1. Note also that B and C are matchings and thus can be also colored each with a color of radius 1.

i) We color $E(G - \mathscr{F})$, B and C with three colors of radius 1. Then, any ℓ-coloring of $G^k[A]$ induces a coloring of the edges of A with the remaining colors, i.e., the ℓ colors of radius k.

ii) We color $E(G - \mathscr{F})$ and C with two colors of radius 1. Then, any ℓ-coloring of $G^k[A]$ (ℓ'-coloring of $G^k[B]$, respectively) induces a coloring of the edges of A (B, respectively) with ℓ (ℓ', respectively) colors of radius k.

iii) We color $E(G - \mathscr{F})$ with one color of radius 1. Then, any ℓ-coloring of $G^k[A]$ (ℓ'-coloring of $G^k[B]$, ℓ''-coloring of $G^k[C]$, respectively) induces a coloring of the edges of A (B, C, respectively) with ℓ (ℓ', ℓ'', respectively) colors of radius k. $\qquad\square$

3 $(1, 1, 1, k, \ldots, k)$-coloring

In this section, we first prove a general upper bound on the required number of colors of radius k in order that all cubic graphs having a 2-factor are $(1, 1, 1, k, \ldots, k)$-colorable. We also prove that all cubic graphs having a 2-factor are $(1, 1, 1, 3, 3)$-colorable and that some cubic graphs having a 2-factor are $(1, 1, 1, 3)$-colorable.

For the case $k = 2$, Payan [21] has shown that one color of radius two is sufficient. Another proof of this result has been given by Fouquet and Vanherpe [11] (Lemma 2.2 is an intermediate step of the proof of this result).

Theorem 3.1 ([11, 21]). *Every cubic graph is $(1, 1, 1, 2)$-colorable.*

Notice that this result is tight since the Petersen graph is not $(1, 1, 1, 3)$-colorable (as it is not 3-edge-colorable and has diameter 2). In order to find similar results for $k \geq 3$, we consider the sequence of integers $(a_k)_{k \geq 2}$ defined by: $a_2 = 2$, $a_3 = 4$ and $a_k = a_{k-1} + 2a_{k-2} + 2$, for $k \geq 4$. Note that this sequence is contained in Sloane Online Encyclopedia of Integers Sequences (A026644 and A167030) and that $a_k = \frac{2^{k+1} - (-1)^{k+1} - 3}{3}$ for $k \geq 2$.

Lemma 3.2. *Let G be cubic graph having a 2-factor, let $\mathscr{F}$ be any 2-factor of G and let $A \subseteq E(\mathscr{F})$ be of type I. For $k \geq 2$, the graph $G^k[A]$, satisfies $\Delta(G^k[A]) \leq a_k$.*

Proof. Let $\mathscr{F}'$ be the subgraph of $\mathscr{F}$ only containing the odd cycles of $\mathscr{F}$. Note that $A \subseteq E(\mathscr{F}')$. Let e be an edge of $\mathscr{F}$ and let D_e^k be the subgraph of $\mathscr{F}'$ induced by the vertices at distance at most $k - 1$ from any extremity of e. For an integer k, let n_k be the maximum number of connected components of D_e^k among all cubic graphs having a 2-factor, all choice of 2-factor in these graphs and all choice of edge e in these 2-factors. In order to prove that $\Delta(G^k[A]) \leq a_k$, we are going to show that $n_k \leq a_k + 1$. Note that several connected components of D_e^k can be included in the same odd cycle of $\mathscr{F}'$.

The number of connected components of D_e^{k+1} which are not in D_e^k is bounded by two times the number of connected components of D_e^k which are not trivial (the connected components containing only one vertex) which is itself bounded by $2n_{k-1}$. We can easily prove that $n_2 \leq 3 = a_2 + 1$ and $n_3 \leq 5 = a_3 + 1$. By induction, we suppose that $n_{k-1} \leq a_{k-1} + 1$ and $n_k \leq a_k + 1$. For $k \geq 2$, we obtain that $n_{k+1} \leq n_k + 2n_{k-1} \leq a_k + 1 + 2(a_{k-1} + 1) \leq a_k + 2a_{k-1} + 3 \leq a_{k+1} + 1$.

Consequently, since each connected component of D_e^k contains either at most one vertex incident with an edge of A or at most two vertices incident with an edge of A in the case this edge lies in D_e^k, we have $\Delta(G^k[A]) \leq n_k - 1 \leq a_k$ (the minus 1 comes from the connected component containing e). $\qquad\square$

3.1 $(1,1,1,3,\ldots,3)$-coloring

In this subsection, we try to minimize the number of required integers 3 in order that all cubic graphs having a 2-factor are $(1,1,1,3,\ldots,3)$-colorable. We give two results about this problem but we are aware that the result of the following theorem can probably be sharpened since we have not been able to find an infinite family of non $(1,1,1,3)$-colorable cubic graphs. However, at the end of the subsection, we exhibit two non $(1,1,1,3)$-colorable bridgeless cubic graphs.

Theorem 3.3. *Every cubic graph having a 2-factor is $(1,1,1,3,3)$-colorable.*

Proof. Let G be a cubic graph. Let $\mathscr{F}$ be a 2-factor of G having a minimum number of odd cycles. Let $\mathscr{F}'$ be the set of odd cycles from $\mathscr{F}$. By Lemma 2.3.i), if there exists a set A of type I in $\mathscr{F}'$ such that $\Delta(G^3[A]) \leq 1$, then G is $(1,1,1,3,3)$-colorable.

We give labels to the vertices of $\mathscr{F}'$ as follows. If a vertex belonging to an odd cycle C from $\mathscr{F}'$ has a neighbor in a different cycle of $\mathscr{F}'$, we label it by $+$, otherwise we label it by $-$. By Proposition 2.1, the two end vertices of any edge of any cycle C of $\mathscr{F}'$ have neighbors in only at most one cycle of $\mathscr{F}'$ other than C. Thus, if consecutive vertices are labeled by $+$ in an odd cycle C from $\mathscr{F}'$, then there exists an unique cycle C' of $\mathscr{F}'$ such that all these vertices have neighbors only in $C \cup C'$.

Observation 1. For each edge $e \in E(G - \mathscr{F})$ having an extremity u in a odd cycle C from $\mathscr{F}'$ the following is true:

i) if u is labeled by $+$, then all edges in $\mathscr{F}'$ at distance at most 2 from e are included in $C \cup C'$, for C' an odd cycle of $\mathscr{F}'$;

ii) if u is labeled by $-$, then all edges in $\mathscr{F}'$ at distance at most 2 from e are included in C.

The previous observation can be easily obtained using the fact that e is only adjacent with edges of $\mathscr{F}$, these edges being themselves adjacent with edges either in the same cycle than they or in $G - \mathscr{F}$.
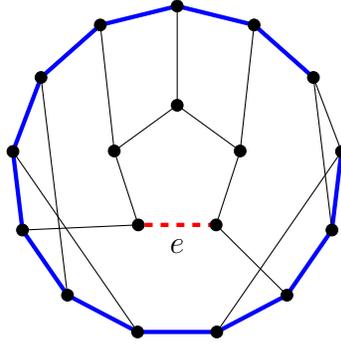
Figure 3: An edge e at distance at most 3 of twelve edges from a 13-cycle (dashed line: e; thick lines: edges at distance at most 3 from e in the 13-cycle).

We will construct A, starting from an empty set, as follows. Since each cycle C of $\mathscr{F}'$ has an odd number of vertices there exist two consecutive vertices both labeled either by $+$ or by $-$ in every cycle of $\mathscr{F}'$. Let u_1 and u_2 be these two adjacent vertices (both labeled either by $+$ or $-$) and suppose that u_0, u_1, u_2, u_3 and u_4 are consecutive vertices of the cycle C (if C contains three vertices then $u_3 = u_0$ and $u_4 = u_1$). For each cycle C of $\mathscr{F}'$, we add to A an edge of C depending on the label of u_1 and u_2.

If u_1 and u_2 are both labeled by $+$ in C, then we add the edge u_1u_2 into A. Note that u_0, u_1, u_2, u_3 are labeled either by $+, +, +, +$, by $-, +, +, +$, by $+, +, +, -$ or by $-, +, +, -$ (the labels are given following the index of u). Consequently, by Observation 1, there exists a cycle C' from $\mathscr{F}'$ such that all edges from $\mathscr{F}'$ at distance at most 3 from u_1u_2 are in $C \cup C'$.

If u_1 and u_2 are both labeled by $-$, then we add the edge u_2u_3 to A. Note that u_1, u_2, u_3, u_4 are labeled either by $-, -, +, +$, by $-, -, -, +$, by $-, -, +, -$ or by $-, -, -, -$. Also in this case, by Observation 1, there exists a cycle C' from $\mathscr{F}'$ such that all edges from $\mathscr{F}'$ at distance at most 3 from u_1u_2 are in $C \cup C'$.

Since there is one edge of A per cycle of $\mathscr{F}'$ we obtain, by construction, that $\Delta(G^3[A]) \leq 1$ and thus $\chi(G^3[A]) \leq 2$. Finally, by Lemma 2.3.i), G is $(1,1,1,3,3)$-colorable. $\qquad\square$

In the following proposition, we prove that cubic graphs of oddness 2 are $(1,1,1,3)$-colorable in the case there are restrictions on the structure of the odd cycles of a 2-factor. Note that Property i) of the following proposition implies that every cubic graph of girth 13 and oddness 2 is $(1,1,1,3)$-colorable.

Proposition 3.4. *Every cubic graph of oddness 2 having a 2-factor containing two odd cycles C_1 and C_2 is $(1,1,1,3)$-colorable in the following cases:*

  i) If C_1 or C_2 is a cycle of length at least 13.

  ii) If $\{C_1, C_2\}$ contains a cycle C of length at least 9 and if there exists at least one edge with one extremity in $\{C_1, C_2\} \setminus C$ and with the other extremity in $G - C$.

iii) If $\{C_1, C_2\}$ contains a cycle C of length at least 5 and if there exists an edge of $\{C_1, C_2\} \setminus C$ with both extremities having no neighbors in C.

Proof. i) Suppose that C_1 has length 13. Let e be an edge from C_2. The edge e is adjacent with two edges not belonging to C_2. Moreover, each of these two edges is possibly at distance at most 2 of at most four edges of C_1. Also, there are possibly two edges at distance 2 from e which have one extremity in C_2 and the other extremity in C_1. Each of these two edges are adjacent with at most two edges of C_1. Thus, e can be at distance at most 3 from at most twelve edges of C_1. Figure 3 illustrates an edge at distance at most 3 from exactly twelve edges. Hence, we can color e and one edge of C_1 at distance at least 4 from e with the color of radius 3 and the remaining uncolored edges with the three colors of radius 1.

ii) Suppose that C_1 has length at least 9 and there exists $xy \in E(G)$ with $x \in V(C_2)$ and $y \in V(G - C_1)$. Let e be an edge of C_2 incident with x. In contrast with Property i), the edges adjacent with e are at distance at most 2 of at most four edges of C_1 (in Property i), it was eight). Thus, e is possibly at distance at most 3 from at most eight edges of C_1. Hence, we can color e and one edge of C_1 at distance at least 4 from e with the color of radius 3 and the remaining uncolored edges with the three colors of radius 1.

iii) Let C_1 be the cycle of length at least 5 and let e be an edge of C_2 with no extremity having a neighbor in C_1. In contrast with Property i), the edges adjacent with e which are not in C_2 cannot be at distance at most 2 of an edge of C_1 (in Property ii), e was at distance at most 2 of eight edges of C_1). Thus, e is possibly at distance at most 3 from at most four edges of C_1. Hence, we can color e and one edge of C_1 at distance at least 4 from e with the color of radius 3 and the remaining uncolored edges with the three colors of radius 1. $\square$

Since the flower snarks of order at least 20 have a 2-factor containing two cycles with one cycle among them of order at least 15, Proposition 3.4.i) implies that flower snarks of order at least 20 are $(1, 1, 1, 3)$-colorable.

Similarly, generalized Petersen graphs of order at least 26 have a 2-factor containing two cycles with one cycle among them of order at least 13. Moreover, generalized Pertersen graphs of order which is a multiple of 4 are 3-edge-colorable. Thus, the generalized Petersen graphs of order at least 24 are $(1, 1, 1, 3)$-colorable.

Note that the Pertersen and Tietze graphs are non $(1, 1, 1, 3)$-colorable since they have oddness 2 and the distance between any two edges in any 2-factor of these graphs is at most 3. By computer we have verified that every cubic graph of order at most 22, except the Petersen graph and the Tietze graph, is $(1, 1, 1, 3)$-colorable.

Corollary 3.5. *The Petersen graph is the only generalized Petersen graph which is not $(1, 1, 1, 3)$-colorable and the Tietze graph is the only flower snark which is not $(1, 1, 1, 3)$-colorable.*

Finally, in the next proposition, we prove that every cubic graph of oddness 4 is also $(1, 1, 1, 3)$-colorable when the odd cycles of the 2-factor are sufficiently large.
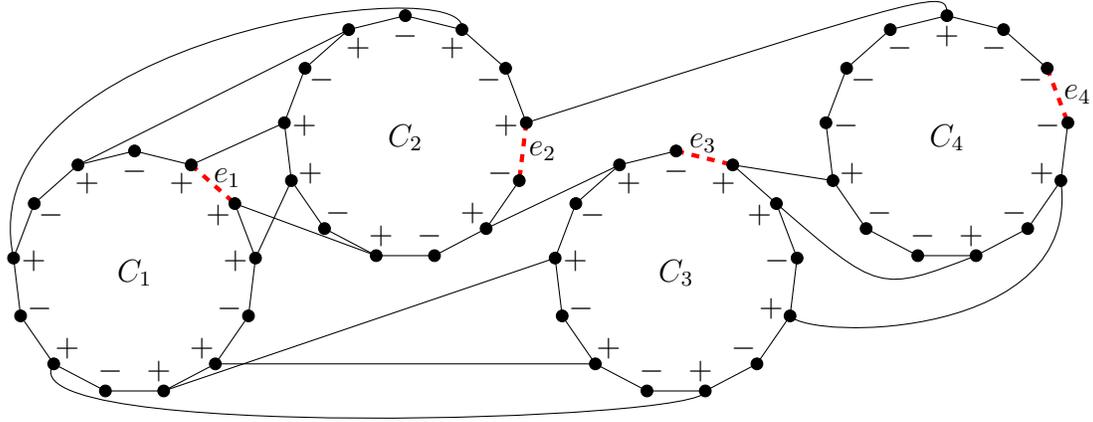
Figure 4: An illustration of the odd cycles $\{C_1, C_2, C_3, C_4\}$ from a 2-factor considered in proof of Proposition 3.6 (dashed lines: edges of A; the edge having end vertices labeled by $-$ in $G - \mathscr{F}$ are omitted in order to simplify the figure).

Proposition 3.6. *Every cubic graph of oddness* 4 *having a* 2*-factor with four odd cycles and with three cycles among them of length at least* 13 *is* $(1, 1, 1, 3)$*-colorable.*

Proof. Let G be a cubic graph of oddness 4 and let $\mathscr{F}'$ be the set containing the four odd cycles from the 2-factor. Let C_1 be the odd cycle having the minimum number of vertices in $\mathscr{F}'$. As in proof of Theorem 3.3, we give labels to the vertices of $\mathscr{F}'$ as follows. If a vertex belonging to C from $\mathscr{F}'$ has a neighbor in a different cycle of $\mathscr{F}'$, we label it by $+$, otherwise we label it by $-$. Figure 4 illustrates a labeling of the vertices of the odd cycles of a 2-factor.

Let e_1 be an edge of C_1 such that there exists a cycle C_2 from $\mathscr{F}'$ such that all edges at distance at most 3 from e_1 in $\mathscr{F}'$ are in $C_1 \cup C_2$. Such an edge can be obtained by proceeding as in the proof of Theorem 3.3. As in the proof of Proposition 3.4.i), e_1 is at distance at most 3 of at most twelve edges of C_2. Let e_2 be an edge of C_2 being at distance at least 4 of e_1. Such an edge exists since C_2 has at least thirteen edges.

First, suppose there exists a cycle C_3 of $\mathscr{F}'$ such that all edges at distance at most 3 from e_2 are in $C_2 \cup C_3$. As previously, by Proposition 3.4.i), e_2 is at distance at most 3 of at most twelve edges of of C_3. Let e_3 be an edge of C_3 being at distance at least 4 of e_2. Also in this case, e_3 is at distance at most 3 of at most twelve edges of C_4, C_4 being the odd cycle of $\mathscr{F}' - (C_1 \cup C_2 \cup C_3)$. Consequently, there exists an edge e_4 of C_4 such that e_4 is at distance at least 4 of e_3. Finally, $A = \{e_1, e_2, e_3, e_4\}$ is a set of type I such that $G^4[A]$ is an empty graph.

Second, by excluding the first case, the two odd cycles C_3 and C_4 of $\mathscr{F}' - (C_1 \cup C_2)$ both contain edges at distance at most 3 from e_2. Figure 4 illustrates the edge e_2 in this case. Let u_1 and u_2 be the extremities of e_2 and suppose u_0, u_1, u_2 and u_3 are consecutive vertices of C_2. By hypothesis and by Proposition 2.1, these vertices are either labeled by $+, -, +, +,$

or by $+,-,+,-$, or by $-,+,-,+$ or by $+,+,-,+$. Up to symmetry, we can suppose that these vertices are either labeled by $+,-,+,+$ or by $+,-,+,-$ (as in Figure 4). Let C_3 be the odd cycle of $\mathscr{F}' - (C_1 \cup C_2)$ containing a neighbor of u_0. By excluding the first case, u_2 and u_3 have no neighbor in C_3. Let v_0 be the neighbor of u_0 in C_3. Suppose v_0, v_1, v_2 and v_3 are consecutive vertices of C_3 and let $e_3 = v_1 v_2$. By Hypothesis v_0 is labeled by $+$. Note that v_0, v_1, v_2 and v_3 are labeled consecutively by $+,-,+,+$ (as in Figure 4), or by $+,-,-,+$, or by $+,-,-,+$ or by $+,-,-,-$. Consequently, there are at most six edges at distance at most 3 from e_2 in C_4 and at most six edges at distance at most 3 from e_3 in C_4. Since C_4 has length at least thirteen, there remains an edge e_4 in C_4 such that e_4 is at distance at least 4 from both e_2 and e_3, as Figure 4 illustrates. Finally, $A = \{e_1, e_2, e_3, e_4\}$ is a set of type I such that $G^4[A]$ is an empty graph and Lemma 2.3.i) allows to conclude. □

We have performed computations on cubic graphs with small order that lead us to believe that all cubic graphs, except two snarks, are $(1,1,1,3)$-colorable.

Conjecture 1. *Every cubic graph different from the Petersen graph and from the Tietze graph is $(1,1,1,3)$-colorable.*

A weaker form of the previous conjecture is to restrict to cubic graphs of oddness 2 or 4.

3.2 $(1,1,1,4,\ldots,4)$-coloring

As in the previous subsection, we try in this subsection to minimize the number of required integers 4 in order that all cubic graphs having a 2-factor are $(1,1,1,4,\ldots,4)$-colorable. We give a first result about this problem, however it is probably not tight since we have not been able to find a non $(1,1,1,4,4)$-colorable cubic graph.

Theorem 3.7. *Every cubic graph having a 2-factor is $(1,1,1,4^5)$-colorable.*

Proof. Let G be a cubic graph. Let $\mathscr{F}$ be 2-factor of G having a minimum number of odd cycles. Let $\mathscr{F}'$ be the set of odd cycles from $\mathscr{F}$. By Lemma 2.3.i) and by Brooks' theorem, it is sufficient to exhibit a set A of type I in $\mathscr{F}$ such that $\Delta(G^4[A]) \leq 4$ in order G to be $(1,1,1,4^5)$-colorable.

As in the proof of Theorem 3.3, we give labels to the vertices of $\mathscr{F}'$ as follows. If a vertex belonging to a cycle C from $\mathscr{F}'$ has a neighbor in a different cycle of $\mathscr{F}'$, we label it by $+$, otherwise we label it by $-$. By Proposition 2.1, the two end vertices of any edge of any cycle C of $\mathscr{F}'$ have neighbors in only at most one cycle of $\mathscr{F}'$ other than C. Thus, if there are consecutive vertices labeled by $+$ in an odd cycle C from $\mathscr{F}'$, then there exists an unique cycle C' of $\mathscr{F}'$ such that all these vertices have neighbors only in $C \cup C'$.

Let u be a vertex of a odd cycle C from $\mathscr{F}'$ and suppose that u_0, u, u_1 are consecutive vertices of C. Let e_0 be the edge incident with u_0 in $G - \mathscr{F}'$ and let e_1 be the edge incident with u_1 in $G - \mathscr{F}'$. We make the following observation in which $D_3^C(u)$ denotes the set of

odd cycles from $\mathscr{F}' - C$ containing a vertex at distance at most 3 from u in the subgraph $G - \{e_0, e_1\}$.

Observation 2. For any vertex u in an odd cycle C from $\mathscr{F}'$ the following is true:

i) if u is labeled by $+$, then $|D_3^C(u)| = 1$;

ii) if u is labeled by $-$, then $|D_3^C(u)| \leq 2$.

Observation 2 can be easily obtained by observing that in the case u is labeled by $+$, the other extremity of the edge incident with u in $G - \mathscr{F}'$ is labeled by $+$ and its neighbors are either labeled by $+$ (and, in this case, are adjacent with vertices of C) or by $-$.

We will construct A as follows. Since each cycle of $\mathscr{F}'$ has an odd number of vertices there exists two consecutive vertices labeled by $+$ or by $-$ in every cycle of $\mathscr{F}'$. For each cycle C of $\mathscr{F}'$, we add an edge of C into A depending on the existence of two consecutive vertices labeled by $+$ and the existence of vertices labeled by $+$.

If there exist two consecutive vertices u_0 and u_1 which are both labeled by $+$ in C, then we add the edge $u_0 u_1$ into A. Let v_0 be a vertex adjacent to u_0 and let v_1 be a vertex adjacent to u_1, v_0 and v_1 being in a cycle C' from $\mathscr{F}' - C$. Let $e_0 = u_0 v_0$ and let $e_1 = u_1 v_1$. By Observation 2, there are no edges at distance less than 4 of either e_0 or e_1 in $\mathscr{F}' - (C \cup C')$. Finally, suppose $u_{-2}, u_{-1}, u_0, u_1, u_2, u_3$ are consecutive vertices of C. The vertices $u_{-2}, u_{-1}, u_0, u_1, u_2$ and u_3 are labeled in the worst case by $+, -, +, +, -, +$. Effectively, if u_{-1} or u_2 is labeled by $+$, then, by Proposition 2.1, its neighbors should be in $C \cup C'$. Thus, in the other cases, there are less vertices among u_{-2}, u_{-1}, u_2 and u_3 which are labeled by $+$ and without having a neighbor in C'. Therefore, $u_0 u_1$ has degree at most 3 in $G^4[A]$.

Suppose now that every vertex of C is labeled by $-$. Let $e = uv$ be an edge of C. As in proof of Theorem 3.3, no edge of C can be at distance less than 4 of an edge of $\mathscr{F}' - C$. By Observation 2 on the two vertices u and v, we obtain that e has degree at most 4 in $G^4[A]$.

By excluding the two previous cases and since C has odd length, there exists four consecutive vertices v_{-2}, v_{-1}, v_0, v_1 labeled by $-, -, +, -$. Let v_2 and v_3 be the vertices such that $v_{-2}, \ldots, v_2, v_3$ are consecutive in C. In this case, we add $v_0 v_1$ into A. Without loss of generality, we can suppose that, by excluding the first case, there are no two consecutive vertices of C labeled by $+$. Thus, $v_{-2}, v_{-1}, v_0, v_1, v_2$ and v_3 are labeled consecutively by $-, -, +, -, +, -$, or by $-, -, +, -, -, +$ or by $-, -, +, -, -, -$. Thus, by Observation 2, $v_0 v_1$ has degree at most 4 in $G^4[A]$. $\square$

We have performed computations on cubic graphs with small order that lead us to believe that all cubic graphs are $(1, 1, 1, 4, 4)$-colorable.

Conjecture 2. *Every cubic graph is $(1, 1, 1, 4, 4)$-colorable.*

Trivially, this conjecture holds for cubic graphs of oddness 2 and a weaker form of the previous conjecture is to restrict the conjecture to cubic graphs of oddness 4. Note that

we have found, performing computations, four bridgeless cubic graphs of order 14 which are $(1, 1, 1, 3)$-colorable but are not $(1, 1, 1, 4)$-colorable.

Also, we state the following open problem which has already been solved for $k \leq 2$ [11].

Question 3.8. *For every integer $k \geq 3$, is it true that there exists a cubic graph of arbitrary large order which is not $(1, 1, 1, k)$-colorable ?*

4 $(1, 1, k, \ldots, k)$-coloring

As for $(1, 1, 1, k, \ldots, k)$-colorings, we first show a general result for arbitrary k. For this, we consider the following two sequences of integers: let $(b_k)_{k \geq 2}$ and $(c_k)_{k \geq 1}$ be sequences of integers defined for $k \geq 2$ by $c_1 = 0$, $c_k = 2^k - c_{k-1}$ and $b_k = \sum_{i=1}^{k} c_i$. The sequence (b_k) is known in OEIS under the number A153772 and has a closed-form formulæ: $b_k = \frac{2^{k+2} + 2(-1)^{k+2} - 6}{3}$.

Lemma 4.1. *Let G be a cubic graph having a 2-factor $\mathscr{F}$ and let $B \subseteq E(\mathscr{F})$ be of type II. For $k \geq 2$, the graph $G^k[B]$ satisfies $\Delta(G^k[B]) \leq b_k$.*

Proof. Let $e \in B$. The proof consists in showing that there are at most b_k edges at distance at most k from e in G. The rest of the proof is divided into three claims.

Claim 1. There are at most 2^{k+1} edges of G at distance k from e.

There are four edges adjacent to e since G is cubic. Suppose now that there are at most 2^k edges at distance k from e. Since each edge at distance k from e is adjacent to at most two edges which are not at distance at most k from e, we obtain that there are at most 2^{k+1} edges at distance $k + 1$ from e.

Claim 2. If any 3-edge-colorable cubic graph G' with any 2-factor $\mathscr{F}'$ of G' only containing even cycles, and any set $B' \subseteq E(\mathscr{F}')$ of type II is such that $\Delta(G'^k[B']) \leq b_k$, then $\Delta(G^k[B]) \leq b_k$.

Remark that each odd cycle of $\mathscr{F}$ contains two adjacent edges of $E(G) \setminus B$. We create a new graph G' by replacing, in each odd cycle of $\mathscr{F}$, one of these two adjacent edges of $E(G) \setminus B$ by a path of length 2, i.e., by two adjacent edges. The vertices of degree 2 created are connected together by a perfect matching (the number of created vertices of degree 2 is even since the number of odd cycles in $\mathscr{F}$ is even). We construct B' from B by adding, for each odd cycle of $\mathscr{F}$, an edge of this cycle which has no neighboring edge of B to B'.

We easily remark that $\Delta(G'^k[B']) \leq b_k$ implies $\Delta(G^k[B]) \leq b_k$.

Claim 3. Suppose G is a 3-edge colorable graph and $\mathscr{F}$ only contains even cycles. There are at most c_k edges at distance k from e in B and at most $2^{k+1} - c_k$ edges at distance k from e in $E(G) \setminus B$, for $c_1 = 0$ and $c_k = 2^k - c_{k-1}$, $k \geq 2$.

Let n_A^k and $n_{E\backslash A}^k$ be the number of edges at distance k from e in B and in $E(G)\backslash B$, respectively. There are four edges adjacent to e, each of these edges not being in B. Thus $n_A^1 = 0$ and $n_{E\backslash A}^1 = 4$. Now suppose by induction that $n_A^k \leq c^k$ and $n_{E\backslash A}^k \leq 2^{k+1} - c_k$. Observe that any edge from $e' \in B$ has no neighbor in B and at most two neighboring edges in $E(G) \backslash B$ which are at greater distance from e than e'. Also, observe that an edge $e'' \in E(G)\backslash B$ has at most one neighbor in B which is at greater distance from e than e'' and at most one neighbor in $E(G) \backslash B$ which is at greater distance from e than e''. Thus we obtain that $n_A^{k+1} \leq n_{E\backslash A}^k \leq 2^{k+1} - c_k = c_{k+1}$ and that $n_{E\backslash A}^{k+1} \leq 2n_A^k + n_{E\backslash A}^k \leq 2c_k + 2^{k+1} - c_k = c_k + 2^{k+1} = 2^{k+2} - 2^{k+1} + c_k = 2^{k+2} - c_{k+1}$.

These three claims together allow to obtain that there are at most b_k edges of A at distance at most k from e. Therefore, the graph $G^k[B]$ satisfies $\Delta(G^k[B]) \leq b_k$. $\quad\square$

Combining this lemma with the ones of Section 2, we obtain the following theorem.

Theorem 4.2. *For any $k \geq 2$, every cubic graph having a 2-factor is $(1,1,k^{a_k+b_k+2})$-colorable and every 3-edge-colorable cubic graph having a 2-factor is $(1,1,k^{b_k+1})$-colorable, where $a_k = \frac{2^{k+1}-(-1)^{k+1}-3}{3}$ and $b_k = \frac{2^{k+2}+2(-1)^{k+2}-6}{3}$.*

Proof. Let G be a cubic graph and $k \geq 2$ be an integer. Let $\mathscr{F}$ be a 2-factor of G. Let $A \subseteq E(\mathscr{F})$ be of type I. By Lemma 3.2, $\Delta(G^k[A]) \leq a_k$. Thus, by Brooks' theorem, $G^k[A]$ is $(a_k + 1)$-colorable. Let $B \subseteq E(\mathscr{F}) \backslash A$ be of type II. By Lemma 4.1, $\Delta(G^k[B]) \leq b_k$. Thus, by Brooks' theorem, $G^k[B]$ is $(b_k + 1)$-colorable.

Therefore, by Lemma 2.3.ii), G is $(1, 1, k^{b_k+a_k+2})$-colorable and even $(1, 1, k^{b_k+1})$-colorable in the case G is 3-edge-colorable (since then, $\mathscr{F}$ can be chosen such that A is empty). $\quad\square$

4.1 $(1, 1, 2, \ldots, 2)$-coloring

The following proposition allows to refine our results for sequences of the form $(1, 1, 2, \ldots, 2)$.

Proposition 4.3. *Let G be a cubic graph of order at least 12. If there exists a 2-factor $\mathscr{F}$ in G, then for any set $B \subseteq E(\mathscr{F})$ of type II, $G^2[B]$ contains no connected component isomorphic to K_5.*

Proof. We first prove that if five edges of B form a connected component isomorphic to K_5 in $G^2[B]$ then these edges lie on the same cycle of $\mathscr{F}$. Suppose to the contrary, that B_5 is a set containing five edges of B forming a K_5 in $G^2[B]$ and that there are, without loss of generality, two cycles C_1 and C_2 of $\mathscr{F}$ such that at least three edges of B_5 are in C_1 and at least one edge of B_5 is in C_2. An edge $e = xy$ of C_2 cannot be at distance 2 of more than two edges of $B_5 \cap C_1$ (the edges of C_1 at distance 2 from e are those who have one endpoint adjacent to x or to y). Hence, e is not adjacent (in $G^2[B]$) to at least one edge of $B_5 \cap C_1$, a contradiction.

Now, we show that if $G^2[B]$ contains a K_5, then G has order 10, which contradicts the hypothesis. Let C be the cycle containing the five edges of B forming a K_5. Note that if an edge of $B \cap C$ has one extremity which is adjacent to a vertex of $G \backslash C$, then this edge

is at distance (in the cycle) at most 2 of at most three other edges from $B \cap C$. Therefore, each vertex of C has its neighbors in C and, consequently, C must have length 10. Since the graph G is connected, G has order 10. $\square$

In any cubic graph G having a 2-factor, we can find a 2-factor containing a minimum number of odd cycles and by Lemma 2.2, there exists, in such 2-factor, a set A of type I such that $G^2[A]$ is an empty graph. Moreover, by virtue of Proposition 4.3 and by Brooks' theorem, every set B of type II in a 2-factor of a cubic graph of order at least 12 satisfies $\chi(G^2[B]) \leq 4$. Consequently, by Lemma 2.3.ii) and by computation on the cubic graphs of order up to 20, we obtain the following result for the sequences containing two times the integer 1 and a bounded number of times the integer 2.

Corollary 4.4. *Every cubic graph having a 2-factor is $(1, 1, 2^5)$-colorable. Every 3-edge-colorable cubic graph having a 2-factor is $(1, 1, 2^4)$-colorable.*

We now present sharper results for graphs with no short cycles. Note that the graphs considered in the two following results are the subcubic graphs. By Euler's formula, there do not exist finite planar cubic graphs of girth at least 6.

Lemma 4.5. *Let G be a subcubic graph having 2-factor $\mathscr{F}$ and let $B \subseteq E(\mathscr{F})$ be of type II. If G is planar and of girth at least 7, then $G^2[B]$ has no triangle and is planar.*

Proof. Let $\mathscr{F}$ be any 2-factor of G and let $B \subseteq E(\mathscr{F})$ be a set of type II. Suppose that $e_1 = x_1 y_1, e_2 = x_2 y_2, e_3 = x_3 y_3$ are three edges of B that form a triangle in $G^2[B]$. Then e_1 is at distance 2 of e_2 and hence one extremity of e_1, say x_1, is adjacent with one extremity of e_2, say x_2, i.e., $x_1 x_2 \in E(G)$. Similarly, y_1 is adjacent with one extremity of e_3, say y_3 and x_3 is adjacent with y_2. But then $x_1, x_2, y_2, x_3, y_3, y_1, x_1$ is a cycle of length six, contradicting the hypothesis on the girth. Consequently, $G^2[B]$ has no triangles. The fact that $G^2[B]$ is planar can be seen by drawing it 'on' a cross-free embedding of the graph G on the plane, putting vertices of $G^2[B]$ on the corresponding edges of G and drawing edges of $G^2[B]$ along the shortest paths between corresponding edges of G. $\square$

Proposition 4.6. *If a subcubic graph G of girth at least 7 is a subgraph of a planar cubic graph G' having a 2-factor, then G is $(1, 1, 2^4)$-colorable. Moreover, if G' is bridgeless, then G is $(1, 1, 2^3)$-colorable.*

Proof. Let G be a subcubic graph of girth at least 7 and let G' be a planar cubic graph having a 2-factor such that G is a subgraph of G'. Let $\mathscr{F}$ be a 2-factor of G' containing a minimum number of odd cycles. By Lemma 2.2, there exists in $\mathscr{F}$ a set A of type I such that $G'^2[A]$ is an empty graph.

Tait [23] has proven that the four color theorem is equivalent to the statement that no snark is planar. Thus, by the four color theorem, G' is 3-edge-colorable in the case G' is bridgeless and the 2-factor $\mathscr{F}$ of G' (containing a minimum number of odd cycles) contains no odd cycles.

If G' contains a bridge (contains no bridges, respectively), let B be a set of type II in $\mathscr{F} - A$ (in $\mathscr{F}$, respectively). Let $B' = B \cap E(G)$. Thus, by Lemma 4.5, $G^2[B']$ has no

triangle and is planar. By the famous Grötzsch's theorem, $G^2[B']$ is 3-colorable. Finally, by Lemma 2.3.ii) G is $(1, 1, 2^4)$-colorable and even $(1, 1, 2^3)$-colorable in the case G' is bridgeless. □

We now show necessary conditions for a cubic graph to be $(1, 1, 2, 2)$-colorable. For this, we first show a relation between $(1, 1, 2, 2)$-coloring and a coloring of the vertices with two colors such that the subgraph induced by each color class has degree exactly one. This relation is used to prove Proposition 4.8. A graph G is said to be *2-matching-colorable* if there exists a partition of $V(G)$ into two sets A_1 and A_2 such that both $G[A_1]$ and $G[A_2]$ are graphs of minimum and maximum degree 1, i.e., matchings. Note that a 2-matching colorable graph is also what is called, in the context of defective coloring, a $(2, 1)$-colorable graph [6], where the list $(2, 1)$ means here that we can use two colors for which each of the color class induce a subgraph of maximum degree 1. Planar graphs have been especially studied in the context of defective coloring [6, 15, 22].

Proposition 4.7. *Let G be a cubic graph. The graph G is 2-matching-colorable if and only if G is $(1, 1, 2, 2)$-colorable.*

Proof. Suppose G is 2-matching-colorable in two set A_1 and A_2. Note that the edges of $G[A_1]$ form a color class of radius 2 in G and the same goes for $G[A_2]$. Moreover, the edges which are not in $G[A_1]$ or $G[A_2]$ form a disjoint union of even cycles. Thus, we can easily color these edges with two colors of radius 1. Therefore, G is $(1, 1, 2, 2)$-colorable.

Now suppose that G is $(1, 1, 2, 2)$-colorable. Let X_1 be the set of vertices incident with an edge colored with the first color of radius 2 and let X_2 be the set of vertices incident with an edge colored with the second color of radius 2. Note that, by definition of color of radius 2, both X_1 and X_2 should induce a graph of maximum degree 1.

We now prove that $X_1 \cap X_2 = \emptyset$ and, afterward, that $X_1 \cup X_2 = V(G)$. First, suppose that there exists a vertex u in $X_1 \cap X_2$. Let e be the edge incident with u which does not have an extremity in X_1 or X_2 and let v be the other extremity of e. By hypothesis, e should be colored with a color of radius 1. The two edges incident with v (different from e) can not be both colored (note that we can not use any color of radius 2 for these two edges) and we obtain that G is not $(1, 1, 2, 2)$-colorable.

Second, suppose that there exists a vertex u not in $X_1 \cup X_2$. Since u is incident with three edges which do not have colors of radius 2, we obtain a contradiction with the fact that G is $(1, 1, 2, 2)$-colorable.

Therefore, we obtain that X_1 and X_2 form a partition of $V(G)$ and that G is 2-matching colorable. □

Proposition 4.8. *Any $(1, 1, 2, 2)$-colorable cubic graph G satisfies the following properties:*

   i) G is 3-edge colorable;
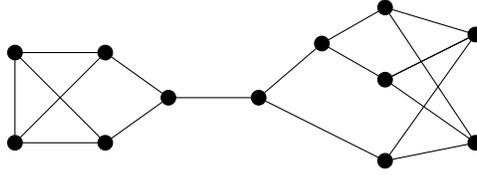
   ii) G has order divisible by four.

Figure 5: The smallest non $(1, 1, 2, 2, 2)$-colorable and non $(1, 2^6)$-colorable cubic graph.

Proof. By Proposition 4.7, G is 2-matching-colorable in two set A_1 and A_2.

i) Note that the edges of $G[A_1] \cup G[A_2]$ form a perfect matching. Note also that the edges with one extremity in A_1 and the other extremity in A_2 form a disjoint union of even cycles. Consequently, G is 3-edge-colorable.

ii) Since G is cubic, we have $|A_1| = |A_2|$. Moreover, by definition of 2-matching colorable, $|A_1|$ is even. Therefore, G has order divisible by four. $\square$

We end this subsection by pointing out that there exist cubic graphs that are not $(1, 1, 2, 2, 2)$-colorable. The smallest such graph, illustrated in Figure 5 has order 12 and has been found by exhaustive search using a computer.

It seems that the results of Proposition 4.6 can be extended to the whole class of cubic graphs. We state this as a conjecture:

Conjecture 3. *Every cubic graph is $(1, 1, 2, 2, 2, 2)$-colorable and every 3-edge-colorable cubic graph is $(1, 1, 2, 2, 2)$-colorable.*

4.2 $(1, 1, 3, \ldots, 3)$-coloring and $(1, 1, 4, \ldots, 4)$-coloring

We finish this section by giving general results about the required number of integers 3 and 4 in order that all cubic graph having a 2-factor are $(1, 1, 3, \ldots, 3)$-colorable and $(1, 1, 4, \ldots, 4)$-colorable.

Proposition 4.9. *Every cubic graph G having a 2-factor is $(1, 1, 3^{11})$-colorable. Moreover, if G is 3-edge-colorable, then G is $(1, 1, 3^9)$-colorable. Also there exists a 3-edge-colorable cubic graph which is not $(1, 1, 3^6)$-colorable*

Proof. By Proposition 4.2, every 3-edge-colorable cubic graph having a 2-factor is $(1, 1, 3^9)$-colorable. By Theorem 3.3, we can color a set of type I with two colors of radius 3. Thus, we obtain that every cubic graph having a 2-factor is $(1, 1, 3^{11})$-colorable.

By exhaustive search, we have found a 3-edge-colorable cubic graph G of 14 vertices such that the line graph of G has diameter 3, i.e., every two edges of G are at distance at most 3. Thus, since a matching of G has size at most 7, we can color at most 14 edges with two colors of radius 1 and we obtain that G is not $(1, 1, 3^6)$-colorable since there remain seven uncolored edges. $\square$

Proposition 4.10. *Every cubic graph G having a 2-factor is $(1, 1, 4^{26})$-colorable. More-over, if G is 3-edge-colorable, then G is $(1, 1, 4^{21})$-colorable. Also there exists a 3-edge-colorable cubic graph which is not $(1, 1, 4^{14})$-colorable.*

Proof. By Proposition 4.2, every 3-edge-colorable cubic graph having a 2-factor is $(1, 1, 4^{21})$-colorable. By Theorem 3.7, we can color a set of type I with five colors of radius 4. Thus, we obtain that every cubic graph having a 2-factor is $(1, 1, 4^{26})$-colorable.

By exhaustive search, we have found a 3-edge-colorable cubic graph G of 30 vertices (G has girth 7) such that the line graph of G has diameter 4, i.e., every two edges of G are at distance at most 4. Thus, since a matching of G has size at most 15, we can color at most 30 edges with two colors of radius 1 and we obtain that G is not $(1, 1, 3^{14})$-colorable since there remain fifteen uncolored edges. □

5 $(1, k, \ldots, k)$-coloring

In this section, we show that for any cubic graph having a 2-factor and any fixed integer k, there is a coloring with only one color of radius 1 and a finite number of colors of radius k.

Theorem 5.1. *For any $k \geq 2$, every cubic graph having a 2-factor is $(1, k^{a_k + 2b_k + 3})$-colorable and every 3-edge-colorable cubic graph having a 2-factor is $(1, k^{2b_k + 2})$-colorable, where $a_k = \frac{2^{k+1} - (-1)^{k+1} - 3}{3}$ and $b_k = \frac{2^{k+3} + 2(-1)^{k+1} - 6}{3}$.*

Proof. Let G be a cubic graph and $k \geq 2$ be an integer. Let $\mathscr{F}$ be any 2-factor of G. Clearly, $E(\mathscr{F})$ can be partitioned into three sets A, B and C such that A is of type I and B, C are of type II. By Lemma 3.2, $\Delta(G^k[A]) \leq a_k$. Thus, by Brooks' theorem, $G^k[A]$ is $(a_k + 1)$-colorable. By Lemma 4.1, $\Delta(G^k[B]) \leq b_k$, and $\Delta(G^k[C]) \leq b_k$. Thus, by Brooks' theorem, $G^k[B]$ and $G^k[C]$ are $(b_k + 1)$-colorable.

Therefore, by Lemma 2.3.iii), G is $(1, k^{2b_k + a_k + 3})$-colorable and even $(1, k^{2b_k + 2})$-colorable in the case G is 3-edge-colorable (by setting $A = \emptyset$). □

We remark that this general result is far from tight at least for small values of k: for $k = 2$ it gives that cubic graphs having a 2-factor are $(1, 2^9)$-colorable, but it is known since a long time that such graphs are (2^{10})-colorable [1].

For the case $k = 2$ with restrictions on the graph, we can prove sharper results.

Proposition 5.2. *If a subcubic graph G of girth at least 7 is a subgraph of a planar cubic graph G' having a 2-factor, then G is $(1, 2^7)$-colorable. Moreover, if G' is bridgeless, then G is $(1, 2^6)$-colorable.*

Proof. Let G be a subcubic graph of girth at least 7 and let G' be a planar cubic graph having a 2-factor such that G is a subgraph of G'. Let $\mathscr{F}$ be a 2-factor of G' containing a minimum number of odd cycles. By Lemma 2.2, there exists in $\mathscr{F}$ a set A of type I such that $G'^2[A]$ is an empty graph.

Tait [23] has proven that the four color theorem is equivalent to the statement that no snark is planar. Thus, by the four color theorem, G is 3-edge-colorable in the case G' is bridgeless and the 2-factor $\mathscr{F}$ of G' (containing a minimum number of odd cycles) contains no odd cycles.

If G' contains a bridge (contains no bridges, respectively), let B and C be two sets of type II forming a partition of $\mathscr{F} - A$ ($\mathscr{F}$, respectively). Let $B' = B \cap E(G)$ and $C' = C \cap E(G)$. Thus, by Lemma 4.5, both $G^2[B']$ and $G^2[C']$ have no triangle and are planar. By the famous Grötzsch's theorem, both $G^2[B']$ and $G^2[C']$ are 3-colorable. Finally, by Lemma 2.3.iii), G is $(1, 2^7)$-colorable and even $(1, 2^6)$-colorable in the case G' is bridgeless.

$\square$

As for the previous section, note that there exist non $(1, 2^6)$-colorable cubic graphs, the smallest one being the graph on 12 vertices depicted in Figure 5.

We give the following results for sequences of type $(1, 3, \ldots, 3)$ or $(1, 4, \ldots, 4)$.

Proposition 5.3. *Every cubic graph G having a 2-factor is $(1, 3^{20})$-colorable and $(1, 4^{47})$-colorable. Moreover, if G is 3-edge-colorable, then G is $(1, 3^{18})$-colorable and $(1, 4^{42})$-colorable. Also there exists two 3-edge-colorable cubic graphs G' and G'' such that G' is not $(1, 3^{13})$-colorable and G'' is not $(1, 4^{29})$-colorable.*

Proof. We can easily prove that G is $(1, 3^{20})$-colorable and even $(1, 3^{18})$-colorable in the case G is 3-edge-colorable using the same arguments than in the proof of Proposition 4.9. Using the same arguments than in the proof of Proposition 4.10, we can prove analogous results for sequences of type $(1, 4, \ldots, 4)$.

The graph G' is the non $(1, 1, 3^6)$-colorable cubic graph of Proposition 4.9 and G'' is the non $(1, 1, 4^{14})$-colorable cubic graph of Proposition 4.10. Since the line graph of G' has diameter 3 and the line graph of G'' has diameter 4 and since the matchings of G' has size at most 7 and the matching of G'' has size at most 15, we obtain that G' is not $(1, 3^{13})$-colorable and that G'' is not $(1, 4^{29})$-colorable.

$\square$

The computations we have made let us think that seven colors of radius 2 are enough in general and less with girth restrictions. We end this section by stating these two open problems.

Question 5.4. *Is it true that all cubic graphs are $(1, 2^7)$-colorable ?*

Question 5.5. *Is it true that all cubic graphs of girth at least 5 are $(1, 2^5)$-colorable ?*

6 $(1, 2, \ldots, k)$-coloring

We finish this paper by proving that there is no integer k such that every subcubic graph is $(1, 2, \ldots, k)$-colorable, i.e., that the line graphs of subcubic graphs have arbitrary large

packing chromatic number. We recall that the packing chromatic number of a graph G is the smallest integer k such that there exists a partition of $V(G)$ into k subsets $\{X_1, \ldots, X_k\}$, each X_i being a set of vertices at pairwise distance at least $i + 1$. Note that it has already been proven that for every fixed k and $g \geq 2k + 2$, almost every cubic graph of girth at least g of sufficiently large order has packing chromatic number greater than k [2].

Adding a leaf on a vertex u of a graph G is an operation that consists in adding a new vertex v and the edge uv.

Let T_1 be the graph $K_{1,3}$ and let T_1' be the graph $K_{1,3}$ for which we have added two leaves on the same vertex of degree 1. By induction on i, let T_i (T_i', respectively), for $i \geq 2$, be the graph constructed from T_{i-1} (T_{i-1}', respectively) by adding two leaves on each vertex u of degree 1.

Proposition 6.1. *For any integer k, there exists an integer N such that T_N is not $(1, 2, \ldots, k)$-colorable.*

Proof. By induction on i, we begin by proving that the diameter of the line graph of T_i is $2i - 1$. For $i = 1$, the diameter of the line graph of T_1 is 1. Suppose that the diameter of the line graph of T_i is $2i - 1$. By definition, we obtain that the diameter of the line graph T_{i+1} is $2i + 1$. Similarly, we can prove that the diameter of the line graph of T_i' is $2i$.

Moreover, by induction on i, we prove that $|E(T_i)| = 3(2^i - 1)$. For $i = 1$, $|E(T_i)| = 3$. Suppose that $|E(T_i)| = 3(2^i - 1)$. It is trivial to note that there are $3(2^i - 1) - 3(2^{i-1} - 1) = 3 \, 2^{i-1}$ vertices of degree 1 in T_i. Thus, there are $2(3 \, 2^{i-1})$ new edges in T_{i+1}. Consequently, $|E(T_{i+1})| = 3(2^i - 1) + 2(3 \, 2^{i-1}) = 3(2^{i+1} - 1)$.

Also, by induction on i, we prove that $|E(T_i')| = 2^{i+2} - 3$. For $i = 1$, $|E(T_i')| = 5$. Suppose that $|E(T_i')| = 2^{i+2} - 3$. It is trivial to note that there are $2^{i+2} - 3 - 2^{i+1} + 3 = 2^{i+1}$ vertices of degree 1 in T_i'. Thus, there are $2(2^{i+1})$ new edges in T_{i+1}'. Consequently, $|E(T_{i+1}')| = 2^{i+2} - 3 + 2(2^{i+1}) = 2^{i+3} - 3$.

Let N be a sufficiently large integer (compared to k). Since the diameter of the line graph of T_i is $2i - 1$, an upper bound on the size of an i-packing (a set of edges at pairwise distance at least i) in T_N, for i an odd integer, converges towards $|E(T_N)|/|E(T_{(i+1)/2})|$. Moreover, since the diameter of the line graph of T_i' is $2i$, an upper bound on the size of an i-packing in T_N, for i an even integer, converges towards $|E(T_N)|/|E(T_{i/2}')|$. Thus, if T_N is $(1, 2, \ldots, k)$-colorable and ϵ is an arbitrary small constant, then

$$\sum_{i=1}^{\lceil k/2 \rceil} (|E(T_N)|/|E(T_i)|) + \sum_{i=1}^{\lfloor k/2 \rfloor} (|E(T_N)|/|E(T_i')|) - \epsilon \geq |E(T_n)|.$$

However, by calculation,

$$\sum_{i=1}^{\lceil k/2 \rceil} (1/|E(T_i)|) + \sum_{i=1}^{\lfloor k/2 \rfloor} (1/|E(T_i')|) \leq \sum_{i=1}^{\infty} 1/(3(2^i - 1)) + \sum_{i=1}^{\infty} 1/(2^{i+2} - 3) < 0.8793 < 1.$$

Thus, we obtain a contradiction and T_N is not $(1, 2, \ldots, k)$-colorable.

$\square$

Since T_N is subcubic for any integer N and since for any integer $n > N$, T_n contains T_N as subgraph, we obtain the following corollary.

Corollary 6.2. *There exist non $(1, 2, \ldots, k)$-colorable subcubic graphs of arbitrary large order for every integer k.*

Note also that there exists non $(1, 2, \ldots, k)$-colorable cubic graphs for every integer k, since we can easily construct a cubic graph containing T_N as subgraph, for every integer N.

Acknowledgments

References

[1] L. D. Andersen, The strong chromatic index of a cubic graph is at most 10, *Discrete Mathematics*, 108(1): 231–252, 1992.

[2] J. Balogh, A. Kostochka, X. Liu, Packing chromatic number of cubic graphs, *Discrete Mathematics*, 341(2): 474–483, 2018.

[3] B. Brešar, S. Klavžar, D. F. Rall, Packing chromatic number of base-3 sierpiński graphs, *Graphs and Combinatorics*, 32(4): 1313–1327, 2016.

[4] B. Brešar, S. Klavžar, D. F. Rall, K. Wash, Packing chromatic number under local changes in a graph, *Discrete Mathematics*, 340(5): 1110–1115, 2017.

[5] G. Brinkmann, J. Goedgebeur, J. Hägglund, K. Markström, Generation and properties of snarks, *Journal of Combinatorial Theory*, Series B, 103(4): 468–488, 2013.

[6] R. Cowen, R. H. Cowen, D. R. Woodall, Defective colorings of graphs in surfaces: Partitions into subgraphs of bounded valency, *Journal of Graph Theory* 10 (2): 187–195, 1986.

[7] J. Fiala, P. A. Golovach, Complexity of the packing coloring problem for trees, *Discrete Appl. Math.* 158(7): 771–778, 2010.

[8] A. S. Finbow, D. F. Rall, On the packing chromatic number of some lattices, *Discrete Appl. Math.* 158(12): 1224–1228, 2010.

[9] J. L. Fouquet, J. L. Jolivet, Strong edge-colorings of graphs and applications to multi-k-gons, *Ars Combinatoria*, 16A: 141–150, 1983.

[10] J. L. Fouquet, J. L. Jolivet, Tools for parsimonious edge-coloring of graphs with maximum degree three, *on arXiv*: 1201.5988v1, 2012.

[11] J. L. Fouquet, J. M. Vanherpe, On parsimonious edge-colouring of graphs with maximum degree three, *Graphs and Combinatorics*, 29(3): 475-487, 2013.

[12] N. Gastineau, O. Togni, S-packing colorings of cubic fraphs, *Discrete Mathematics* 339(10): 2461–2470, 2016.

[13] N. Gastineau, H. Kheddouci, O. Togni, Subdivision into i-packings and S-packing chromatic number of some lattices, *Ars Mathematica Contemporanea* 9:331–354, 2015.

[14] N. Gastineau, P. Holub, O. Togni, On the packing chromatic number of subcubic outerplanar graphs, *arXiv*:1703.05023 [cs.DM], 2017.

[15] W. Goddard, Acyclic colorings of planar graphs, *Discrete Mathematics*, 91(1): 91–94, 1991.

[16] W. Goddard, H. Xu, The S-packing chromatic number of a graph, *Discuss. Math. Graph Theory*, 32: 795–806, 2012.

[17] W. Goddard, H. Xu, A note on S-packing colorings of lattices, *Discrete Appl. Math.*, 166: 255–262, 2014.

[18] W. Goddard, S. M. Hedetniemi, S. T. Hedetniemi, J. M. Harris, D. F. Rall, Broadcast chromatic numbers of graphs, *Ars Combinatoria*, 86: 33–49, 2008.

[19] H. Hocquard, M. Montassier, A. Raspaud, P. Valicov, On strong edge-colouring of subcubic graphs, *Discrete Appl. Math.*, 161(16): 2467–2479, 2013.

[20] R. Lukoťka, E. Máčajová, J. Mazák, M. Škoviera, Small snarks with large oddness, *The Electronic Journal of Combinatorics*, 22(1): #P1.51, 2015.

[21] C. Payan, Sur quelques problèmes de couvertures et de couplages en combinatoire. Thèse d'état (in french), 1977.

[22] K. S. Poh, On the linear vertex-arboricity of a planar graph, *Journal of Graph Theory*, 14(1):73–75, 1990.

[23] P. G. Tait, Remarks on the colourings of maps, *Proc. R. Soc. Edinburgh*, 10:729, 1880.