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# ON COMPLETE INTERSECTIONS IN VARIETIES WITH FINITE-DIMENSIONAL MOTIVE

ROBERT LATERVEER, JAN NAGEL, AND CHRIS PETERS

ABSTRACT. Let  $X$  be a complete intersection inside a variety  $M$  with finite dimensional motive and for which the Lefschetz-type conjecture  $B(M)$  holds. We show how conditions on the niveau filtration on the homology of  $X$  influence directly the niveau on the level of Chow groups. This leads to a generalization of Voisin's result. The latter states that *if  $M$  has trivial Chow groups and if  $X$  has non-trivial variable cohomology parametrized by  $c$ -dimensional algebraic cycles, then the cycle class maps  $A_k(X) \rightarrow H_{2k}(X)$  are injective for  $k < c$* . We give variants involving group actions which lead to several new examples with finite dimensional Chow motives.

## 1. INTRODUCTION

**1.1. Background.** Let  $X$  be a smooth, complex projective variety of dimension  $d$ . While the cohomology ring<sup>1</sup>  $H^*(X)$  is well understood, this is far from true for the Chow ring  $A^*(X)$ , the ring of algebraic cycles on  $X$  modulo rational equivalence. The two are linked through the *cycle class map*

$$A^*(X) \rightarrow H^{2*}(X), \quad \gamma \mapsto [\gamma].$$

If this map is injective we say that  $X$  has *trivial Chow groups*. If this is not the case, the kernel  $A_{\text{hom}}^*(X)$ , the "homologically trivial" cycles, then can be investigated through the *Abel-Jacobi map*

$$A_{\text{hom}}^*(X) \rightarrow J^*(X)$$

with kernel  $A_{\text{AJ}}^*(X)$ , the "Abel-Jacobi trivial" cycles. If  $X$  is a curve, Abel's theorem tells us that  $A_{\text{AJ}}^1(X) = 0$ .

The interplay between Hodge theoretic aspects of cohomology and cycles became apparent through the fundamental work of Bloch and Srinivas [8] as complemented by [25, 36]. They investigate the consequences for the Chow groups and cohomology groups of  $X$  if the class  $\delta \in A^d(X \times X)$  of the diagonal  $\Delta \subset X \times X$  admits a decomposition into summands having support on lower dimensional varieties. This clarifies the role of the so-called *coniveau filtration*  $N^\bullet H^*(X)$  in cohomology which takes care of cycle classes supported on varieties of varying dimensions. Charles Vial [45] discovered a variant which works better in homology which he called the *niveau filtration*  $\tilde{N}^\bullet H_*(X)$ . We introduce a *refined niveau filtration* on homology  $\hat{N}^\bullet H_*(X)$  which is compatible with polarizations. The precise definitions are given below in

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<sup>1</sup>See the conventions about the notation at the end of the introduction.

Sect. 2.4. Suffices to say that we have inclusions  $\widehat{N}^\bullet H_*(X) \subseteq \widetilde{N}^\bullet H_*(X) \subseteq N^\bullet H_*(X)$  with equality everywhere if the Lefschetz conjecture  $B$  is true for all varieties. Conjecture  $B$  is recalled below in Section 2.2.

Note that the Künneth formula  $\delta = \sum_{k=0}^{2d} \pi_k$ , with  $\pi_k \in H^{2d-k}(X) \otimes H^k(X) = H^k(X)^* \otimes H^k(X)$ , can be interpreted as an identity inside the ring of endomorphisms of  $H^*(X)$ . Since  $\delta \in H^{2d}(X \times X)$  acts as the identity on  $H^*(X)$ , in  $\text{End } H^*(X)$  one thus obtains the (cohomological) *Künneth-decomposition*

$$\text{id} = \sum_{k=1}^{2d} \pi_k, \quad \pi_k \in \text{End } H^*(X) \text{ a projector with } \pi_k|_{H^j(X)} = \delta_{jk} \cdot \text{id}.$$

The projectors are mutually orthogonal, that is  $\pi_j \circ \pi_k = 0$  if  $j \neq k$ . Moreover, the Künneth decomposition is by construction compatible with Poincaré duality and so is called *self-dual*; in other words  $\pi_k$  is the transpose of  $\pi_{2d-k}$  for all  $k < d$ .

Even if the *Künneth components*  $\pi_k$  are classes of algebraic cycles, their sum need not give a decomposition of the diagonal. If this is the case, and if, moreover, these give a self-dual decomposition of the identity in  $\text{End } A^*(X)$  by mutually orthogonal projectors, one speaks of a (self-dual) *Chow–Künneth decomposition*, abbreviated as ”CK-decomposition”. Its existence has been conjectured by Murre [32], and it has been established in low dimensions and a few other cases.

One would like to have a refined CK-decomposition which takes into account the coniveau filtration or the (refined) niveau filtration, since then the conclusions of [8] et. al. can be applied. This is related to the validity of the standard conjecture  $B(X)$  as reviewed in Section 2.2.

**1.2. Set up and results.** Following Voisin [49, 50], we consider complete intersections  $X$  of dimension  $d$  inside a given smooth complex variety projective variety  $M$  and we ask about the relations between the Chow groups of  $M$  and  $X$ . On the level of cohomology this is a consequence of the classical Lefschetz theorems: apart from the ”middle” cohomology  $H^d(X)$  the cohomology of  $X$  is completely determined by  $H^*(M)$ , while for the middle cohomology one has a direct sum splitting

$$H^d(X) = H_{\text{fix}}^d(X) \oplus H_{\text{var}}^d(X)$$

into fixed cohomology  $H_{\text{fix}}^d(X) = i^* H^d(M)$  and its orthogonal complement  $H_{\text{var}}^d(X)$  under the cupproduct pairing. Here  $i : X \hookrightarrow M$  is the inclusion, and  $i^* : H^d(M) \rightarrow H^d(X)$  is injective.

For this to have consequences on the level of Chow groups, it seems natural to assume that  $M$  has trivial Chow groups. This is the point of view of Voisin in [50]. Her main result uses the notion of a subspace  $H \subset H^k(X)$  ”being parametrized by  $c$ -dimensional algebraic cycles” [50, Def. 0.3] which is slightly stronger than demanding that  $H \subset \widehat{N}^c H^k(X)$ , where  $\widehat{N}$  is our refined version of Vial’s filtration. A comparison of our filtration with Vial’s is given in Section 3.2. See in particular Remark 4.7. We can now state Voisin’s main result from [50]:

**Theorem.** *Assume that  $M$  has trivial Chow groups and that  $X$  has non-trivial variable cohomology parametrized by  $c$ -dimensional algebraic cycles. Then the cycle class maps  $A_k(X) \rightarrow H_{2k}(X)$  are injective for  $k < c$ .*

Our idea is to replace the condition of  $M$  having trivial Chow groups by finite dimensionality of the motive of  $M$  – which conjecturally is true for all varieties.<sup>2</sup> The main idea which makes this operational is the following nilpotency result (=Theorem 2.8): if  $r$  is the codimension of  $X$  in  $M$ , a degree  $r$  correspondence which restricts to a cohomologically trivial degree zero correspondence on  $X$  is nilpotent as a correspondence on  $X$ .

The second ingredient is due to Voisin [49, Proposition 1.6]: a degree  $d$  cohomologically trivial relative correspondence can be modified in a controlled way such that the new relative correspondence is fiberwise *rationaly* equivalent to zero.

Given these inputs, the argument leading to our results now runs as follows. First we make use of the refined niveau filtration by way of Propositions 4.5 and 4.8 to find relative correspondences that decompose the diagonal in *homology* in the way we want. To the difference we apply the Voisin result. This provides first of all information on the level of the *Chow groups of the fibers* and, secondly, allows us to apply the nilpotency result. Writing this out gives strong variants of the above theorem of Voisin. These have been phrased in homology rather than cohomology because, as mentioned before, Vial’s filtration and ours behave better in the homological setting. One of our main results can be paraphrased as follows.

**Theorem** (=Theorem 5.6). *Suppose that  $B(M)$  holds, that the Chow motive of  $M$  is finite-dimensional and that  $H_k(M) = N^{\lfloor \frac{k+1}{2} \rfloor} H_k(M)$  for  $k \leq d$ . Suppose  $H_d^{\text{var}}(X) \neq 0$ , and that for some positive integer  $c$  we have  $H_d^{\text{var}}(X) \subset \hat{N}^c H_d(X)$ . Then  $A_k^{\text{hom}}(X) = 0$  if  $k < c$  or  $k > d - c$ .*

Voisin’s result is a direct consequence: by [43, Theorem 5] varieties with trivial Chow groups have finite dimensional motive and conjecture  $B$  holds for them as well and the condition  $H_k(M) = N^{\lfloor \frac{k+1}{2} \rfloor} H_k(M)$  holds since  $M$  has trivial Chow groups. Surprisingly, if we apply Vial’s result [42], we find that if the condition in the above theorem holds for  $c = \lfloor \frac{d}{2} \rfloor$ , then  $h(X)$  itself also has finite dimension and up to motives of curves and Tate twists is a direct factor of  $h(M)$  (Corollary 5.7).

The known examples of finite dimensional motives are all directly related to curves, which very much limits the search for examples. However, inside the realm of motives we can use other projectors besides the identity, namely those that come from group actions. In Section 6, we have formulated variants of the main result involving actions of a finite abelian group, say  $G$ . Then, even if the level of the Hodge-niveau filtration on variable cohomology is too big to apply our main theorems, there might be a  $G$ -character space which has the correct Hodge-level. Provided the (generalized) Hodge conjecture holds, which is automatically the case in dimensions  $\leq 2$ , this then ensures the desired condition on the niveau filtration. In Section 7 we construct examples where this is the case and for which one of the group variants of the main theorem can be successfully applied. These examples all yield new finite dimensional motives because of the above mentioned result of Vial.

We have given several types of examples:

- a threefold of general type with  $p_g = q = 0$ ,
- hypersurfaces in abelian threefolds, including the Burniat-Inoue surfaces,
- hypersurfaces in a product of a hyperelliptic curve and certain types of K3 surfaces,

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<sup>2</sup>See [33] for background on Chow motives.

- hypersurfaces in threefolds that are products of three curves, one of which is hyperelliptic,
- odd-dimensional complete intersections of 4 quadrics – generalizing the Bardelli example [3].

For simplicity we have only considered involutions since then all invariants can easily be calculated, but it will be clear that the method of construction allows for many more examples of varieties admitting all kinds of finite abelian groups of automorphisms.

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**Notation.** Varieties will be defined over  $\mathbf{C}$  (except for Appendix B, where we consider algebraic varieties and motives over a field  $k$ ). We use  $H^*, H_*$  for the (co)homology groups with  $\mathbf{Q}$ -coefficients and likewise we write  $A^*, A_*$  for the Chow groups with  $\mathbf{Q}$ -coefficients.

The category of Chow motives (over a field  $k$ ) is denoted by  $\text{Mot}_{\text{rat}}(k)$ , the category of *covariant* homological motives by  $\text{Mot}_{\text{hom}}(k)$  and the category of numerical motives  $\text{Mot}_{\text{num}}(k)$ . For a smooth projective manifold  $X$ , we let  $h(X) \in \text{Mot}_{\text{rat}}(k)$  be its Chow motive.

We denote the integer part of a rational number  $a$  by  $[a]$ .

## 2. PRELIMINARIES

**2.1. Correspondences.** If  $X$  and  $Y$  are projective varieties with  $X$  irreducible of dimension  $d_X$ , a correspondence of degree  $p$  is an element of

$$\text{Corr}_p(X, Y) := A_{d_X+p}(X \times Y).$$

A degree  $p$  correspondence  $\gamma$  induces maps

$$\gamma_* : A_k(X) \rightarrow A_{k+p}(Y), \quad \gamma^* : H_k(X) \rightarrow H_{k+2p}(Y).$$

If, moreover,  $X$  and  $Y$  are smooth projective, we have correspondences of cohomological degree  $p$ , i.e., elements

$$\gamma \in \text{Corr}^p(Y, X) := A^{d_Y+p}(Y \times X),$$

which induce

$$\gamma^* : A^k(Y) \rightarrow A^{k+p}(X), \quad \gamma_* : H^k(Y) \rightarrow H^{k+2p}(X).$$

**Definition 2.1.** Let  $\gamma \in \text{Corr}_p(X, X) = A_{d+p}(X \times X)$  be a self-correspondence of degree  $p$  where  $d = d_X$ .

- (1) Let  $Z$  be smooth and equi-dimensional. We say that  $\gamma$  *factors through  $Z$  with shift  $i$*  if there exist correspondences  $\alpha \in \text{Corr}_i(Z, X)$  and  $\beta \in \text{Corr}_{-j}(X, Z)$  ( $i - j = p$ ) such that  $\gamma = \alpha \circ \beta$  and  $d - (i + j) = \dim Z$ .
- (2) We say that  $\gamma$  is *supported on  $V \times W$*  if

$$\gamma \in \text{Im} (A_{d+p}(V \times W) \xrightarrow{(i \times j)_*} A_{d+p}(X \times X))$$

where  $i : V \rightarrow X$  and  $j : W \rightarrow X$  are inclusions of subvarieties of  $X$ .

The usefulness of these concepts follows from the following evident results.

- Lemma 2.2.** (1) *If a correspondence  $\gamma \in \text{Corr}_0(X, X)$  factors through  $Z$  with shift  $c$ , then  $\gamma$  and  ${}^t\gamma$  act trivially on  $A_j(X)$  for  $j < c$  or  $j > d - c$ .*
- (2) *If a correspondence  $\gamma \in \text{Corr}_0(X, X)$  is supported on  $V \times W \subset X \times X$ , then  $\gamma$  acts trivially on  $A_j(X)$  for  $j < \text{codim } V$  or  $j > \dim W$  and  ${}^t\gamma$  acts trivially on  $A_j(X)$  for  $j < \text{codim } W$  or  $j > \dim V$ .*

**2.2. Standard conjecture  $B(X)$ .** Let  $X$  be a smooth complex projective variety of dimension  $d$ , and  $h \in H^2(X)$  the class of an ample line bundle. The hard Lefschetz theorem asserts that the map

$$L_X^{d-k}: H_{2d-k}(X) \rightarrow H_k(X)$$

obtained by cap product with  $h^{d-k}$  is an isomorphism for all  $k < d$ . One of the standard conjectures asserts that the inverse isomorphism is algebraic:

**Definition 2.3.** Given a variety  $X$ , we say that  $B_k(X)$  holds if the isomorphism

$$\Lambda^{d-k} = (L^{d-k})^{-1}: H_k(X) \xrightarrow{\cong} H_{2d-k}(X)$$

is induced by a correspondence. We say that the *Lefschetz standard conjecture  $B(X)$  holds* if  $B_k(X)$  holds for all  $k < d$ .

*Remark 2.4.* The Lefschetz (1,1) theorem implies that  $B_k(X)$  holds if  $k \leq 1$  and hence it holds for curves and surfaces. It is stable under products and hyperplane sections [22, 23] and so, in particular, it is true for complete intersections in products of projective spaces. It is known that  $B(X)$  moreover holds for the following varieties:

- abelian varieties [22, 23];
- threefolds not of general type [40];
- hyperkähler varieties of  $K3^{[n]}$ -type [12];
- Fano varieties of lines on cubic hypersurfaces [30, Corollary 6];
- $d$ -dimensional varieties  $X$  which have  $A_k(X)$  supported on a subvariety of dimension  $k + 2$  for all  $k \leq \frac{d-3}{2}$  [42, Theorem 7.1];
- $d$ -dimensional varieties  $X$  which have  $H_k(X) = N^{\lfloor \frac{k}{2} \rfloor} H_k(X)$  for all  $k > d$  [43, Theorem 4.2].

Below we shall use the following well known implication of  $B(X)$ .

**Proposition 2.5** ([22, Thm. 2.9]). *Suppose that  $B(X)$  holds. Then the Künneth projectors are algebraic, i.e., there exist correspondences  $\pi_k \in \text{Corr}_0(X, X)$  such that  $\pi_{k*} |_{H_j(X)} = \delta_{kj} \cdot \text{id}$  and  $\Delta_X \sim_{\text{hom}} \sum_k \pi_k$ .*

Refinements will be stated below in Section 2.4.

**2.3. Finite dimensional motives and nilpotence.** We refer to [1], [16], [21], [33] for the definition of a Chow motive and its dimension. We also need the concept of a *motive of abelian type*, by definition a Chow motive  $M$  for which some twist  $M(n)$  is a direct summand of the motive of a product of curves.

A crucial property of varieties with finite-dimensional motive is the nilpotence theorem.

**Theorem 2.6** (Kimura [21]). *Let  $X$  be a smooth projective variety with finite-dimensional motive. Let  $\Gamma \in \text{Corr}_0(X, X)$  be a correspondence which is numerically trivial. Then there exists a nonnegative integer  $N$  such that  $\Gamma^{\circ N} = 0$  in  $\text{Corr}_0(X, X)$ .*

Actually, the nilpotence property (for all powers of  $X$ ) could serve as an alternative definition of finite-dimensional motive, as shown by a result of Jannsen [19, Corollary 3.9]. Conjecturally, any variety has finite-dimensional motive [21]. We are still far from knowing this, but at least there are quite a few non-trivial examples:

*Remark 2.7.* The following varieties are known to have a finite-dimensional motive:

- varieties dominated by products of curves [21] as well as varieties of dimension  $\leq 3$  rationally dominated by products of curves [44, Example 3.15];
- $K3$  surfaces with Picard number 19 or 20 [38];
- surfaces not of general type with vanishing geometric genus [14, Theorem 2.11] as well as many examples of surfaces of general type with  $p_g = 0$  [37, 51];
- Hilbert schemes of surfaces known to have finite-dimensional motive [10];
- Fano varieties of lines in smooth cubic threefolds, and Fano varieties of lines in smooth cubic fivefolds [29];
- generalized Kummer varieties [53, Remark 2.9(ii)];
- 3-folds with nef tangent bundle [44, Example 3.16]), as well as certain 3-folds of general type [46, Section 8];
- varieties  $X$  with Abel-Jacobi trivial Chow groups (i.e.  $A_{AJ}^k X = 0$  for all  $k$ ) [43, Theorem 4];
- products of varieties with finite-dimensional motive [21].

*Remark.* It is worth pointing out that up till now, all examples of finite-dimensional Chow motives happen to be of abelian type. On the other hand, “many” motives are known to lie outside this subcategory, e.g. the motive of a general hypersurface in  $\mathbf{P}^3$  [2, Remark 2.34].

The following result is a kind of “weak nilpotence” for subvarieties of a variety  $M$  with finite-dimensional motive; any correspondence that comes from  $M$  and is numerically trivial turns out to be nilpotent.

**Proposition 2.8.** *Let  $M$  be a smooth projective variety with finite-dimensional Chow motive and let  $X \subset M$  be a smooth projective subvariety of codimension  $r$ . For any correspondence  $\Gamma \in \text{Corr}_r(M, M)$  with the property that the restriction*

$$\Gamma|_X \in \text{Corr}_0(X, X)$$

*is homologically trivial, there exists a nonnegative integer  $N$  such that*

$$(\Gamma|_X)^{\circ N} = 0 \text{ in } \text{Corr}_0(X, X).$$

*Proof.* Put  $L = i_* \circ i^* \in \text{Corr}_{-c}(M, M)$  and  $T = \Gamma \circ L \in \text{Corr}_0(M, M)$ . We have

$$\Gamma|_X = (i \times i)^*(\Gamma) = i^* \circ \Gamma \circ i_*.$$

By induction on  $k$  one shows that

$$(1) \quad \Gamma|_X^{k+1} = i^* \circ T^k \circ \Gamma \circ i_*$$

for all  $k \geq 0$ . As

$$\begin{aligned} T^2 &= \Gamma \circ i_* \circ i^* \circ \Gamma \circ i_* \circ i^* \\ &= \Gamma \circ i_* \circ \Gamma_X \circ i^*, \end{aligned}$$

$T^2$  is homologically trivial. Hence  $T^2$  is nilpotent by [21], say  $T^{2\ell} = 0$ . Hence  $\Gamma_X$  is nilpotent of index  $N = 2\ell + 1$  by (1).  $\square$

#### 2.4. Coniveau and niveau filtration.

**Definition 2.9** (Coniveau filtration [7]). Let  $X$  be a smooth projective variety of dimension  $d$ . The  $j$ -th level of the *coniveau filtration* on cohomology (with  $\mathbf{Q}$ -coefficients) is defined as the subspace generated by the classes supported on subvarieties  $Z$  of dimension  $\leq d - j$ :

$$N^j H^k(X) = \sum_Z \text{Im}(i_* : H_Z^k(X) \rightarrow H^k(X)).$$

This gives a decreasing filtration on  $H^k(X)$ . We may instead use smooth varieties  $Y$  of dimension exactly  $d - j$  provided we use degree  $j$  correspondences from  $Y$  to  $X$ : such a correspondence sends  $Y$  to a cycle  $Z$  of dimension  $\leq d - j$  in  $X$  and all cycles can be obtained in this way. When we rewrite this in terms of homology we get

$$N^j H_k(X) = \sum_{Y, \gamma} \text{Im}(\gamma_* : H_k(Y) \rightarrow H_k(X)),$$

where  $Y$  is smooth projective of dimension  $k - j$  and  $\gamma \in \text{Corr}_0(Y, X)$ .

Since the  $j$ -th level of the filtration consists of the classes supported on varieties of dimension  $k - j$ , the filtration stops beyond  $k/2$ : a variety of dimension  $< k/2$  has no homology in degrees  $\geq k$ :

$$0 = N^{\lfloor \frac{k}{2} \rfloor + 1} H_k(X) \subset N^{\lfloor \frac{k}{2} \rfloor} H_k(X) \subset \dots \subset N^1 H_k(X) \subset N^0 H_k(X) = H_k(X).$$

*Remark.* Under Poincaré duality one has an identification  $N^j H^k(X) = N^{d-k+j} H_{2d-k}(X)$ .

Vial [45] introduced the following variant of the coniveau filtration:

**Definition 2.10** (Niveau filtration). Let  $X$  be a smooth projective variety. The *niveau filtration* on homology is defined as

$$\tilde{N}^j H_k(X) = \sum \text{Im}(\gamma^* : H_{k-2j}(Z) \rightarrow H_k(X)),$$

where the sum is taken over all smooth projective varieties  $Z$  of dimension  $k - 2j$ , and all correspondences  $\gamma \in \text{Corr}_j(Z \times X)$ .

*Remark 2.11.* The idea behind this definition is that one should be able to lower the dimension of the variety  $Y$  appearing in Definition 2.9 using the Lefschetz standard conjecture. By Hard Lefschetz we have an isomorphism  $\Lambda_Y^j : H_{k-2j}(Y) \xrightarrow{\cong} H_k(Y)$  and by the Lefschetz hyperplane theorem a surjection  $\iota_* : H_{k-2j}(Z) \rightarrow H_{k-2j}(Y)$  with  $Z = Y \cap H_1 \dots \cap H_j$  a complete intersection



of  $Y$  with  $j$  general hyperplanes. Hence there is a surjective map  $\iota_* \circ \Lambda_Y^j : H_{k-2j}(Z) \rightarrow H_k(Y)$  which is algebraic if  $B_{k-2j}(Y)$  holds and thus  $N^j H_k(X) = \tilde{N}^j H_k(X)$ .

This discussion also shows that

- $\tilde{N}^j H_k(X) \subset N^j H_k(X)$
- $\tilde{N}^j H_k(X) = N^j H_k(X)$  if  $k - 2j \leq 1$ .

**2.5. On variable and fixed cohomology.** Let  $M$  be a smooth projective variety of dimension  $d + r$  and  $i : X \hookrightarrow M$  a smooth complete intersection of dimension  $d$ . Let us assume  $B(M)$  so that the operator  $\Lambda^r$  on  $H_*(M)$  is induced by an algebraic cycle  $\Lambda_M^r$  on  $M \times M$ . Set

$$\pi^{\text{fix}}(X) := i^* \Lambda_M^r i_*, \quad \pi^{\text{var}}(X) = \Delta - \pi^{\text{fix}}(X).$$

Recall that setting

$$\begin{aligned} H_d^{\text{fix}}(X) &= \text{Im}(i^* : H_{d+2r}(M) \rightarrow H_d(X)), \\ H_d^{\text{var}}(X) &= \ker(i_* : H_d(X) \rightarrow H_d(M)), \end{aligned}$$

one has a direct sum decomposition

$$H_d(X) = H_d^{\text{fix}}(X) \oplus H_d^{\text{var}}(X),$$

which is orthogonal with respect to the intersection product. We claim the following result.

**Lemma 2.12.** *The operators  $\pi^{\text{fix}}(X)$  and  $\pi^{\text{var}}(X)$  are homological projectors which give the projection of the total cohomology onto  $H^{\text{fix}}(X)$ , respectively  $H^{\text{var}}(X) = H_d^{\text{var}}(X)$ .*

*Proof.* We first observe that  $i_* : H_*(X) \rightarrow L^r H_*(M)$  since  $i_* H^{\text{fix}}(X) = i_* \circ i^* H(M) = L^r H(M)$ . On the image of  $L$  the two operators  $L$  and  $\Lambda$  are inverses. So, since<sup>3</sup>  $i^* \circ i_* = L^r$ , we find

$$\begin{aligned} (i^* \circ \Lambda^r \circ i_*)^2 &= i^* \circ \Lambda^r \circ i_* \circ i^* \circ \Lambda^r \circ i_* \\ &= i^* \circ \Lambda^r \circ L^r \circ \Lambda^r \circ i_* \\ &= i^* \circ \Lambda^r \circ i_*, \end{aligned}$$

i.e.  $\pi^{\text{fix}}$  is indeed a projector, and so is  $\pi^{\text{var}}$ . These projectors define a splitting on cohomology given by

$$z = i^* \Lambda^r i_* z + (z - i^* \Lambda^r i_* z).$$

On the image of  $i_*$  the two operators  $L$  and  $\Lambda$  commute and are each others inverse and so

$$\begin{aligned} i_*(z - i^* \Lambda^r i_* z) &= i_* z - L^r \Lambda^r i_* z \\ &= i_* z - i_* z = 0 \end{aligned}$$

which shows that  $\pi^{\text{var}}$  indeed gives the projection onto variable homology and so  $\pi^{\text{fix}}$  projects onto the fixed cohomology.  $\square$

*Remark 2.13.* The degree zero correspondences  $\pi^{\text{fix}}$  and  $\pi^{\text{var}}$  are not necessarily projectors on the level of Chow groups, although one can show that finite-dimensionality of  $h(M)$  and  $B(M)$  can be used to modify these correspondences in such a way that they become projectors. For what follows we do not need this.

<sup>3</sup>In fact this is only true up to a multiplicative constant but changing  $\Lambda^r$  accordingly corrects this.

## 3. NIVEAU FILTRATIONS AND POLARISATIONS

**3.1. Polarisation.** Recall that for  $k \leq d = \dim X$  we have the Lefschetz decomposition

$$H^k(X) = \bigoplus_r L^r H_{\text{pr}}^{k-2r}(X).$$

Following [52, p. 77] we define a polarisation  $Q_X$  on  $H^k(X)$  as follows. Given  $a, b \in H^k(X)$ , write  $a = \sum_r L^r a_r, b = \sum_r L^r b_r$  and define

$$Q_X(a, b) = \sum_r (-1)^{\frac{k(k-1)}{2} + r} \langle L^{d-k+2r} a_r, b_r \rangle$$

where

$$\langle , \rangle : H^{2d-k+2r}(X) \otimes H^{k-2r}(X) \rightarrow H^{2d}(X) \cong \mathbf{Q}$$

denotes the cup product. As the Lefschetz decomposition is  $Q_X$ -orthogonal, we can rewrite this in the following form. Let  $p_r : H^k(X) \rightarrow L^r H_{\text{pr}}^{k-2r}(X)$  be the projection, and define

$$s_X = \sum_r (-1)^{\frac{k(k-1)}{2} + r} L^r \circ p_r.$$

Then  $Q_X(a, b) = \langle L^{d-k}(a), s_X(b) \rangle$ .

When we translate this to homology we obtain a polarisation  $Q_X$  on  $H_k(X)$  ( $k \leq d$ ) given by

$$Q_X(a, b) = \langle a, \Lambda^{d-k}(s_X(b)) \rangle$$

where  $s_X$  is (up to sign) the alternating sum of the projections  $p_r : H_k(X) \rightarrow L^r H_{k+2r}^{\text{pr}}(X)$  to the primitive homology (dual to primitive cohomology).

**Lemma 3.1.** *If  $B_\ell(X)$  holds for  $\ell \leq 2 \dim X - k - 2$  the operator  $s_X \in \text{End}(H_k(X))$  is algebraic.*

*Proof.* See [11, Lemma 7] or [45, Lemma 1.7] □

**3.2. Modified niveau filtration.** We start by a discussion of adjoint correspondences. This material is treated from a cohomological point of view in [13, section 4.2].

**Definition 3.2.** Let  $X$  and  $Y$  be smooth projective varieties of dimension  $d_X, d_Y$ . Let  $\gamma \in \text{Corr}_j(X, Y)$ .

(i) We say that  $\gamma$  admits a  $k$ -adjoint if there exists  $\gamma^{\text{adj}} \in \text{Corr}_{-j}(Y, X)$  such that

$$Q_Y(\gamma_*(a), b) = Q_X(a, \gamma_*^{\text{adj}}(b))$$

for all  $a \in H_{k-2j}(X), b \in H_k(Y)$ .

(ii) We say that  $\gamma$  admits an adjoint if it admits a  $k$ -adjoint for all  $k$ .

**Proposition 3.3.** *If the standard conjectures  $B(X)$  and  $B(Y)$  hold, every correspondence  $\gamma \in \text{Corr}(X, Y)$  admits an adjoint.*

*Proof.* Let  $\gamma \in \text{Corr}_j(X, Y)$  and consider the map

$$\gamma_* : H_k(X) \rightarrow H_{k+2j}(Y).$$

As  $B(X)$  and  $B(Y)$  hold, the operators  $s_X$  and  $s_Y$  are algebraic by Lemma 3.1. As  $s_X$  and  $s_Y$  commute with the Lambda operator, we obtain

$$\begin{aligned} Q_Y(\gamma_*(a), b) &= \langle \gamma_*(a), \Lambda_Y^{d_Y-k-2j}(s_Y(b)) \rangle \\ &= \langle a, {}^t\gamma_*(\Lambda_Y^{d_Y-k-2j}(s_Y(b))) \rangle \\ &= \langle a, s_X(\Lambda_X^{d_X-k}(s_X(L_X^{d_X-k}({}^t\gamma_*(\Lambda_Y^{d_Y-k-2j}(s_Y(b)))))) \rangle. \end{aligned}$$

Hence

$$\gamma^{\text{adj}} = s_X \circ L_X^{d_X-k} \circ {}^t\gamma \circ \Lambda_Y^{d_Y-k-2j} \circ s_Y$$

is an adjoint of  $\gamma$ .  $\square$

To use the existence of an adjoint, we need a linear algebra lemma (cf. [47, Lemma 5],[45, Lemma 1.6]).

**Lemma 3.4.** *Let  $H$  and  $H'$  be finite-dimensional  $\mathbf{Q}$ -vector spaces equipped with non degenerate bilinear forms  $Q : H \times H \rightarrow \mathbf{Q}$  and  $Q' : H' \times H' \rightarrow \mathbf{Q}$ . Suppose that there exist linear maps*

$$\alpha : H' \rightarrow H, \quad \beta : H \rightarrow H'$$

*such that*

- (a)  $\alpha$  is surjective;
- (b)  $Q'|_{\text{Im}(\beta \times \beta)}$  is non degenerate;
- (c)  $Q(\alpha(x), y) = Q'(x, \beta(y))$  for all  $x \in H', y \in H$ .

*Then  $\alpha \circ \beta : H \rightarrow H$  is an isomorphism.*

*Proof.* As  $H$  is finite-dimensional, it suffices to show that  $\ker(\alpha \circ \beta) = 0$ . Suppose that  $y \in \ker(\alpha \circ \beta)$ . Then  $\beta(y) \in \ker(\alpha) \cap \text{Im}(\beta)$ . By (c) we have

$$0 = Q(\alpha(\beta(y)), z) = Q'(\beta(y), \beta(z))$$

for all  $z \in H$ , hence  $\beta(y) = 0$  by condition (b). This gives

$$0 = Q'(x, \beta(y)) = Q(\alpha(x), y)$$

for all  $x \in H'$  and since  $\alpha$  is surjective we obtain  $y = 0$ .  $\square$

**Corollary 3.5.** *Suppose that  $\gamma : \text{Corr}_j(Y, X)$  admits an adjoint. Consider the map  $\gamma_* : H_{k-2j}(Y) \rightarrow H_k(X)$ . Then  $\gamma_* \circ \gamma_*^{\text{adj}} : H_k(X) \rightarrow H_k(X)$  induces an isomorphism*

$$\gamma_* \circ \gamma_*^{\text{adj}} : \text{Im}(\gamma_*) \xrightarrow{\sim} \text{Im}(\gamma_*).$$

*Proof.* Apply the previous Lemma with  $H' = H_k(X)$ ,  $\alpha = \gamma_*$ ,  $\beta = \gamma_*^{\text{adj}}$  and  $H = \text{Im}(\gamma_*) \subseteq H_k(X)$ . Condition (a) is satisfied by construction, (b) by Hodge theory (Hodge-Riemann bilinear relations) and (c) by the adjoint condition.  $\square$

**Definition 3.6.** The *modified niveau filtration*  $\widehat{N}^\bullet$  is defined by

$$\widehat{N}^j H_k(X) = \sum \text{Im}(\gamma_* : H_{k-2j}(Z) \rightarrow H_k(X)),$$

where the sum runs over all pairs  $(Z, \gamma)$  such that  $Z$  is smooth projective of dimension  $k - 2j$  and such that  $\gamma \in \text{Corr}_j(Z, X)$  admits a  $k$ -adjoint.

We have

$$\widehat{N}^j H_k(X) \subseteq \widetilde{N}^j H_k(X) \subseteq N^j H_k(X).$$

The filtrations  $N^\bullet$  and  $\widetilde{N}^\bullet$  are compatible with the action of correspondences. The filtration  $\widehat{N}^\bullet$  is compatible with correspondences that admit an adjoint.

**Proposition 3.7.** *Let  $\gamma \in \text{Corr}_j(X, Y)$ . If  $B(X)$  and  $B(Y)$  hold then we have  $\gamma_* \widehat{N}^c H_k(X) \subseteq \widehat{N}^{c+j} H_{k+2j}(Y)$ .*

*Proof.* There exist a smooth projective variety  $Z$  and a correspondence  $\lambda \in \text{Corr}_c(Z, X)$  such that  $\lambda$  admits an adjoint and

$$\widehat{N}^c H_k(X) = \text{Im } \lambda_* : H_{k-2c}(Z) \rightarrow H_k(X).$$

We have

$$\lambda_* \widehat{N}^c H_k(X) = \text{Im} (\gamma \circ \lambda)_* : H_{k-2c}(Z) \rightarrow H_{k+2j}(Y)$$

The image is contained in  $\widehat{N}^{c+j} H_{k+2j}(Y)$  since  $\gamma$  admits an adjoint by Proposition 3.3 and  $(\gamma \circ \lambda)^{\text{adj}} = \lambda^{\text{adj}} \circ \gamma^{\text{adj}}$ .  $\square$

#### 4. ON KÜNNETH DECOMPOSITIONS

**Definition 4.1.** Let  $X$  be a smooth projective variety.

- (1) We say that  $X$  admits a *refined Künneth decomposition* if there exist correspondences  $\pi_{i,j} \in \text{Corr}_0(X, X)$  such that
  - $\Delta_X \sim_{\text{hom}} \sum_{i,j} \pi_{i,j}$
  - $(\pi_{i,j})_* |_{\text{Gr}_N^q H_p(X)} = \begin{cases} \text{id} & \text{if } (p, q) = (i, j) \\ 0 & (p, q) \neq (i, j). \end{cases}$
  - $\pi_{i,j} = 0$  if and only if  $\text{Gr}_N^j H_i(X) = 0$ .
- (2) We say that  $X$  admits a *refined Chow–Künneth decomposition* if in addition the  $\pi_{i,j}$  are projectors and  $\Delta_X \sim_{\text{rat}} \sum_{i,j} \pi_{i,j}$ .
- (3) We say that  $X$  admits a refined Künneth (or Chow–Künneth) decomposition *in the strong sense* if  $\pi_{i,j}$  factors with shift  $j$  through a smooth, projective variety  $Z_{i,j}$  of dimension  $i - 2j$  for all  $i$  and  $j$ .

*Remark 4.2.* By [45, Prop. 1.4] there exists a  $Q_X$ -orthogonal splitting

$$H^*(X) = \bigoplus_{i,j} \text{Gr}_N^j H_i(X).$$

The variety  $X$  admits a refined Künneth decomposition if this decomposition lifts to the category  $\text{Mot}_{\text{hom}}(k)$  of homological motives. It admits a refined Chow–Künneth decomposition if the decomposition lifts to the category  $\text{Mot}_{\text{rat}}(k)$  of Chow motives.

In an analogous way one can define refined Künneth (Chow–Künneth) decompositions with respect to the filtrations  $\widetilde{N}^\bullet$  and  $\widehat{N}^\bullet$ .

The proof of the following result is a reformulation of the proof of [45, Thm. 1] in terms of the modified niveau filtration.

**Proposition 4.3.** *If  $B(X)$  holds, there exists a refined Künneth decomposition in the strong sense with respect to the filtration  $\widehat{N}^\bullet$ .*

*Proof.* Conjecture  $B(X)$  implies that the Künneth components are algebraic, i.e., there exist correspondences  $\pi_i \in \text{Corr}_0(X, X)$  such that  $(\pi_i)_*|_{H_j(X)} = \delta_{ij} \cdot \text{id}$ . By Proposition 3.7 the proof of [45, Prop. 1.4] goes through for the filtration  $\widehat{N}^\bullet$ , and we obtain a  $Q_X$ -orthogonal splitting

$$H^*(X) = \bigoplus_{i,j} \text{Gr}_{\widehat{N}}^j H_i(X).$$

The aim is to construct correspondences  $\pi_{i,j} \in \text{Corr}_0(X, X)$  that induce this decomposition. This is done by descending induction on  $j$ . If  $j > i/2$  we take  $\pi_{i,j} = 0$ . Suppose that the correspondences  $\pi_{i,k}$  have been constructed for  $k > j$ . As before there exist  $Z$ , smooth of dimension  $i - 2j$ , and  $\gamma \in \text{Corr}_j(Z, X)$  such that

$$\widehat{N}^j H_i(X) = \text{Im}(\gamma_* : H_{i-2j}(Z) \rightarrow H_i(X)).$$

By replacing  $\gamma$  with  $\pi_{i \circ \gamma}$  if necessary, we may assume that  $\gamma_*|_{H_\ell(Z)} = 0$  if  $\ell \neq i - 2j$ . The correspondence  $\pi = \pi_i - \sum_{k>j} \pi_{i,k}$  induces the projection  $\widehat{N}^j H_i(X) \rightarrow \text{Gr}_{\widehat{N}}^j H_i(X)$ . Put  $\gamma' = \pi \circ \gamma$ . By construction

$$\gamma'_* : H_{i-2j}(Z) \rightarrow \text{Gr}_{\widehat{N}}^j H_i(X)$$

is surjective. As  $B(X)$  holds,  $\pi$  admits an adjoint by Proposition 3.3. By definition  $\gamma$  admits an adjoint, hence  $\gamma' = \pi \circ \gamma$  admits an adjoint and the correspondence  $T = \gamma' \circ (\gamma')^{\text{adj}}$  induces an isomorphism

$$\varphi = T_* : \text{Gr}_{\widehat{N}}^j H_i(X) \rightarrow \text{Gr}_{\widehat{N}}^j H_i(X)$$

by Corollary 3.5. By the Cayley–Hamilton theorem there exists a polynomial expression  $\psi = P(\varphi)$  such that  $\psi \circ \varphi = \text{id}$ . Put  $U = \psi(T)$  and define  $\pi_{i,j} = U \circ T$ . As  $T_* = \varphi$  and  $U_* = \psi$  we have

$$\begin{aligned} (\pi_{i,j})_*|_{\text{Gr}_{\widehat{N}}^j H_i(X)} &= \text{id} \\ (\pi_{i,j})_*|_{\text{Gr}_{\widehat{N}}^q H_p(X)} &= 0 \text{ if } (p, q) \neq (i, j). \end{aligned}$$

By construction  $\pi_{i,j}$  factors with shift  $j$  through a smooth projective variety of dimension  $i - 2j$  and  $\pi_{i,j} = 0$  if and only if  $\text{Gr}_{\widehat{N}}^j H_i(X) = 0$ .  $\square$

**Corollary 4.4.** *If  $B(X)$  holds and  $H_k(X) \subseteq \widehat{N}^c H_k(X)$ , then there exists  $\pi'_k \in \text{Corr}_0(X, X)$  such that  $\pi_k \sim_{\text{hom}} \pi'_k$  and such that  $\pi'_k$  factors with shift  $c$  through a smooth projective variety  $Z$  as in Definition 2.1.*

*Proof.* By Proposition 4.3 we obtain a decomposition

$$\pi_k = \sum_j \pi_{k,j}.$$

with respect to the filtration  $\widehat{N}^\bullet$ . As  $H_k(X) \subseteq \widehat{N}^c H_k(X)$  we have  $\pi_{k,j} = 0$  for all  $j < c$ , and the result follows.  $\square$

The Corollary can be generalised to the following setting. Suppose that there exists  $\pi_k \in \text{Corr}_0(X, X)$  such that  $(\pi_k)_*|_{H_\ell(X)} = \delta_{k\ell} \cdot \text{id}$ . If  $\pi \in \text{Corr}_0(X, X)$  satisfies

$$\begin{aligned} \pi \circ \pi &\sim_{\text{hom}} \pi \\ \pi \circ \pi_k &\sim_{\text{hom}} \pi_k \circ \pi \sim_{\text{hom}} \pi \end{aligned}$$

the motive  $(X, \pi)$  is a direct factor of  $(X, \pi_k)$  in  $\text{Mot}_{\text{hom}}(k)$ .

**Corollary 4.5.** *Suppose that  $B(X)$  holds and that  $\pi \in \text{Corr}_0(X, X)$  is a correspondence as above. Let  $H_\pi = \text{Im}(\pi) \subseteq H_k(X)$  be the sub-Hodge structure defined by  $\pi$ . If  $H_\pi \subseteq \widehat{N}^c H_k(X)$  there exists a correspondence  $\pi' \sim_{\text{hom}} \pi$  such that  $\pi'$  factors with shift  $c$  through a smooth projective variety  $Z$  as in Definition 2.1.*

*Proof.* The proof of Proposition 4.3 shows that we have a decomposition  $\pi_k = \sum_j \pi_{k,j}$  in  $\text{Mot}_{\text{hom}}(k)$ . Hence

$$\pi = \pi_k \circ \pi = \sum_j \pi_{k,j} \circ \pi.$$

Suppose that there exists  $j_0 < c$  such that  $\pi_{k,j_0} \circ \pi \neq 0$ . Then there exists  $x \in H_k(X)$  such that  $\pi_{k,j_0}(\pi(x)) \neq 0$ . Hence  $H_\pi \cap \text{Im}(\pi_{k,j_0}) \neq 0$ . This contradicts the hypothesis  $H_\pi \subseteq \widehat{N}^c H_k(X)$  since  $\pi_{k,j_0}|_{\widehat{N}^c H_k(X)} = 0$ .  $\square$

This result implies a modification of [26, Cor. 3.4, Lemma 3.5] that we need later on.

**Corollary 4.6.** *Same assumptions about  $M$  and  $X$ . Suppose that  $H_d^{\text{var}}(X) \subset \widehat{N}^c H_d(X)$ . Then  $\pi^{\text{var}} \sim_{\text{hom}} \widetilde{\pi}^{\text{var}}$  where  $\widetilde{\pi}^{\text{var}} \in \text{Corr}^0(X, X)$  factors through a smooth projective variety  $Z$  with shift  $c$  in the sense of Definition 2.1.*

*Remark 4.7.* The condition  $H_d(X) \subset \widehat{N}^c H_d(X)$  may be replaced by Voisin's condition of "being parametrized by algebraic cycles of codimension  $c$ " [50, Def. 0.3]. Voisin's condition implies that

$$\gamma_* \circ {}^t \gamma_* : H_d(X) \rightarrow H_d(X)$$

is a multiple of the identity. Our condition implies that there exists an adjoint  $\gamma^{\text{adj}}$  such that  $\gamma_* \circ \gamma^{\text{adj}}_*$  is an isomorphism with an algebraic inverse (see Corollary 3.5 and the proof of Proposition 4.5). This weaker result suffices for our purposes.

**Proposition 4.8.** *Suppose that  $B(X)$  holds and that for every smooth projective variety  $Z$  of dimension  $k - 2j$  the condition  $B_\ell(Z)$  holds if  $\ell \leq k - 2j - 2$ . Then  $\widetilde{N}^j H_k(X) = \widehat{N}^j H_k(X)$ .*

*Proof.* It suffices to show that for every pair  $(Z, \gamma)$  as in Definition 3.6,  $\gamma$  admits a  $k$ -adjoint. This follows directly from Lemma 3.1.  $\square$

**Corollary 4.9.** *We have  $\tilde{N}^j H_k(X) = \hat{N}^j H_k(X)$  if  $k - 2j \leq 3$ . In particular, if  $H_k(X) = N^{\lfloor \frac{k}{2} \rfloor} H_k(X)$  the filtrations  $\tilde{N}$  and  $\hat{N}$  on  $H_k(X)$  coincide with the coniveau filtration. This is true unconditionally on  $H_k(X)$ ,  $k \leq 3$ . If the conjecture  $B(M)$  holds, all three filtrations are equal on  $H_k(X)$  for  $k \leq 4$ .*

*Remark.* The condition  $B_\ell(Z)$  in Proposition 4.8 is needed to obtain an algebraic correspondence that induces  $s_Z$ . If  $H \subset H_d(X)$  is a sub-Hodge structure such that there exists a smooth projective variety  $Z$  of dimension  $d - 2c$  such that  $H_{d-2c}^{\text{pr}}(Z) \rightarrow H$  is surjective then this condition is not needed and we have  $H \subset \hat{N}^c H_d(X)$ . We present an example below.

**Example 4.10.** Let  $X \subset \mathbf{P}^{d+1}$  be a smooth hypersurface of degree  $d+1$ . Let  $Z = F_1(X)$  be the Fano variety of lines contained in  $X$ . If  $X$  is general then  $Z$  is smooth of dimension  $d - 2$  and the incidence correspondence induces a surjective map (cylinder homomorphism)

$$\gamma_* : H_{d-2}^{\text{pr}}(Z) \rightarrow H_d^{\text{pr}}(X);$$

see [31, Thm. (5.34)]. Hence  $H_d^{\text{pr}}(X) \subset \hat{N}^1 H_d(X)$  by the previous remark.

Concerning the existence of a refined Chow–Künneth decomposition (in the strong sense) for the filtrations  $N^\bullet$ ,  $\tilde{N}^\bullet$  and  $\hat{N}^\bullet$  we have the following.

**Proposition 4.11.** *Let  $X$  be a smooth projective variety over  $\mathbf{C}$  such that  $B(X)$  holds and  $h(X)$  is finite dimensional. Then*

- (i) *There exists a refined Chow–Künneth decomposition in the strong sense for the filtration  $\hat{N}^\bullet$ .*
- (ii) *There exists a refined Chow–Künneth decomposition in the strong sense for*
  - $\tilde{N}^\bullet$  *if  $\dim X \leq 5$ ;*
  - $N^\bullet$  *if  $\dim X \leq 3$ .*

*Proof.* By Proposition 4.3 there exists a refined Künneth decomposition in the strong sense for the filtration  $\hat{N}^\bullet$ . If  $h(X)$  is finite-dimensional the ideal

$$\ker A_d(X \times X) \rightarrow H_{2d}(X \times X)$$

is nilpotent, and the refined Künneth decomposition lifts to  $\text{Mot}_{\text{rat}}(k)$  by a lemma of Jannsen [18]. This proves part (i). Part (ii) follows from the comparison between the filtrations:  $\tilde{N}^j H_i(X) = \hat{N}^j H_i(X)$  if  $j - 2i \leq 3$  (Corollary 4.9) and  $N^j H_i(X) = \tilde{N}^j H_i(X)$  if  $j - 2i \leq 1$ .  $\square$

*Remark 4.12.* Part (ii) is due to Vial [45]. The assumption  $\dim X \leq 5$  can be replaced by the conditions of Proposition 4.8.

*Remark 4.13.* Using Proposition 4.11, the main result of [27] can be extended to arbitrary dimension, provided one replaces Vial’s filtration  $\tilde{N}^\bullet$  in the statement of [27, Theorem 3] by the filtration  $\hat{N}^\bullet$ .

## 5. THE MAIN RESULTS

The setup that we consider in this section is the following. Let  $M$  be a smooth projective variety of dimension  $d + r$ . Let  $L_1, \dots, L_r$  be very ample line bundles on  $M$ , and let  $f : \mathcal{X} \rightarrow B$  denote the family of all smooth complete intersections of dimension  $d$  defined by sections of  $E = L_1 \oplus \dots \oplus L_r$ . We write  $X_b = f^{-1}(b)$ . The next result plays a major role in deriving the main results. It uses the assumption that the  $L_j$  are very ample in a crucial way.

**Proposition 5.1** (Voisin [50]). *Suppose that for general  $b \in B$  one has that  $X_b$  has nontrivial variable homology in degree  $d$ . Let  $\mathcal{D}$  be a codimension- $d$  cycle on  $\mathcal{X} \times_B \mathcal{X}$  with the property that*

$$\mathcal{D}|_{X_b \times X_b} = 0 \text{ in } H_{2d}(X_b \times X_b).$$

*Then there exists a codimension- $d$  cycle  $\gamma$  on  $M \times M$  such that*

$$\mathcal{D}|_{X_b \times X_b} - \gamma|_{X_b \times X_b} = 0 \text{ in } A_d(X_b \times X_b)$$

*for all  $b \in B$ .*

*Proof.* We want to sketch a proof of Voisin's original result [50, Proposition 1.6] since we want to point out where the assumptions are used. Consider the blow up  $\widetilde{M \times M}$  of the diagonal and the natural quotient map  $\mu : \widetilde{M \times M} \rightarrow M^{[2]}$  to the Hilbert scheme of zero-dimensional subschemes of  $M$  of length two. Set  $\mathbf{P} = \mathbf{P}H^0(X, E)$  and as in [50, Lemma 1.3] introduce

$$I_2(E) := \{(s, y) \in \mathbf{P} \times \widetilde{M \times M} \mid s|_{\mu(y)} = 0\}.$$

Next, consider the blow up of  $\mathcal{X} \times_B \mathcal{X}$  along the relative diagonal:

$$p : \widetilde{\mathcal{X} \times_B \mathcal{X}} \rightarrow \mathcal{X} \times_B \mathcal{X}.$$

Observe that  $\widetilde{\mathcal{X} \times_B \mathcal{X}}$  is Zariski-open in  $I_2(E)$  and so it makes sense to restrict cycles on  $I_2(E)$  to the fibers  $X_b \times X_b$  of  $\mathcal{X} \times_B \mathcal{X} \rightarrow B$ . Very ampleness of the  $L_j$  implies that  $I_2(E) \rightarrow M \times M$  is a projective bundle and hence its cohomology can be expressed in terms of cohomology coming from  $\widetilde{M \times M}$  and a tautological class. Assume now that

$$\exists R \in A^d(I_2(E)) \text{ with } R|_{\widetilde{X_b \times X_b}} \sim_{\text{hom}} 0.$$

Voisin shows that this implies the existence of a codimension- $d$  cycle  $\gamma$  on  $M \times M$  and an integer  $k$  such that

$$(p_b)_*(R|_{\widetilde{X_b \times X_b}}) = k\Delta_{X_b \times X_b} + \gamma|_{X_b \times X_b} \text{ in } A_d(X_b \times X_b)$$

The first summand acts on all of homology, while the second summand, by construction, acts only on the fixed homology. So the assumption that there is some variable homology implies that  $k = 0$  and so the cycle  $\gamma$  is homologous to zero. To prove the above variation, suppose we are given  $\mathcal{D}$  of codimension  $d$  on  $\mathcal{X} \times_B \mathcal{X}$  as above. As  $\widetilde{\mathcal{X} \times_B \mathcal{X}} \subset I_2(E)$  is Zariski open, there exists a codimension- $d$  cycle  $R$  on  $I_2(E)$  such that  $R|_{\widetilde{\mathcal{X} \times_B \mathcal{X}}} = p^*\mathcal{D}$ . Then we have

$$R|_{\widetilde{X_b \times X_b}} = p^*\mathcal{D}|_{\widetilde{X_b \times X_b}} = (p_b)^*(\mathcal{D}|_{X_b \times X_b}) = 0 \text{ in } H_{2d}(\widetilde{X_b \times X_b})$$



for all  $b \in B$ , where  $p_b: \widetilde{X_b \times X_b} \rightarrow X_b \times X_b$  denotes the blow-up of the diagonal. Hence, if we apply Voisin's original proposition to this cycle  $R$ , we get the desired conclusion.  $\square$

**Theorem 5.2.** *Notation as above. Suppose that  $B(M)$  holds and the Chow motive of  $M$  is finite-dimensional. Assume that for a general  $b \in B$  the fiber  $X_b$  has non-trivial variable homology:*

$$H_d(X_b)^{\text{var}} \neq 0,$$

and that for some nonnegative integers  $c, e$ , with  $e < d$  we have

$$H_k(X_b) = \widehat{N}^c H_k(X_b) \text{ for all } k \in \{e+1, \dots, d\}.$$

Then for any  $b \in B$

$$\text{Niveau}(A_k(X_b)) \leq e - k \text{ for all } k < \min\{d - e, c\},$$

i.e., there exists a subvariety  $Y_b \subset X_b$  of dimension  $e$  such that  $A_k(Y_b) \rightarrow A_k(X_b)$  is surjective.

*Proof. Step 1.* We first construct a homological decomposition of the diagonal of  $X_b$

$$\Delta_{X_b} \sim_{\text{hom}} \Delta_{\text{left}} + \Delta_{\text{mid}} + \Delta_{\text{right}} \text{ in } H_{2d}(X_b \times X_b),$$

where the right hand side are self-correspondences of  $X$  of degree 0,  $\Delta_{\text{right}} = {}^t\Delta_{\text{left}}$  and  $\Delta_{\text{mid}}$  factors with shift  $c$  through a smooth variety  $Z$ .

This is done as follows. As conjecture  $B$  is stable by hyperplane sections (see Remark 2.4), the complete intersections  $X_b$  satisfy  $B(X_b)$  and hence by Proposition 2.5 there are correspondences  $\pi_j \in \text{Corr}^0(X_b, X_b)$ ,  $j = 0, \dots, 2d$  inducing the corresponding homological Künneth projectors. By Proposition 4.5, for  $k \in \{e+1, \dots, d\}$  we have that  $\pi_k(X_b) \sim_{\text{hom}} \pi'_k(X_b)$ , a projector that factors through a variety with shift  $c$  as in Definition 2.1. Now set

$$\begin{aligned} \Delta_{\text{left}} &= \sum_{k \leq e} \pi_k(X_b) \\ \Delta_{\text{right}} &= {}^t\Delta_{\text{left}} \\ \Delta_{\text{mid}} &= \sum_{k=e+1}^{2d-e-1} \pi'_k(X_b). \end{aligned}$$

*Step 2.* We spread out the fiberwise correspondences  $\Delta_{\text{left}}, \Delta_{\text{right}}, \Delta_{\text{mid}}$  to the family of hyper-surfaces

$$\mathcal{X} \rightarrow B,$$

using Voisin's argument in the form of propositions A.1 and A.2. This gives a homological decomposition of the relative diagonal, in the sense that there exist  $\mathcal{Y} \subset \mathcal{X}$  of relative dimension  $d$  and a family  $\mathcal{Z} \rightarrow B$  of relative dimension  $d - 2c$ , and codimension- $d$  cycles

$$\Pi_{\text{left}}, \quad \Pi_{\text{right}}, \quad \Pi_{\text{mid}}$$

on  $\mathcal{X} \times_B \mathcal{X}$  such that  $\Pi_{\text{left}}, \Pi_{\text{right}}$  have support on  $\mathcal{Y} \times_B \mathcal{X}$ , resp. on  $\mathcal{X} \times_B \mathcal{Y}$ , and  $\Pi_{\text{mid}}$  factors through  $\mathcal{Z} \rightarrow B$  such that for any  $b \in B$ , restriction gives back the diagonal:

$$\left( \Pi_{\text{left}} + \Pi_{\text{mid}} + \Pi_{\text{right}} \right) \Big|_{X_b \times X_b} = \Delta_{X_b} \text{ in } H_{2d}(X_b \times X_b).$$

*Step 3.* We upgrade this to rational equivalence using properties of  $M$ . So we consider the difference

$$\mathcal{D} := \Delta_{\mathcal{X}} - \Pi_{\text{left}} - \Pi_{\text{mid}} - \Pi_{\text{right}},$$

a relative correspondence with the property that

$$\mathcal{D}|_{X_b \times X_b} = 0 \text{ in } H_{2d}(X_b \times X_b),$$

for all  $b \in B$ . To upgrade this to rational equivalence we applying the key Proposition 5.1 to  $\mathcal{D}$ . We find a codimension- $d$  cycle  $\gamma$  on  $M \times M$  such that

$$\mathcal{D}|_{X_b \times X_b} - \gamma|_{X_b \times X_b} = 0 \text{ in } \text{Corr}_0(X_b \times X_b),$$

for all  $b \in B$ . The crucial point is that the restriction  $\gamma|_{X_b \times X_b} \in \text{Corr}_0(X_b \times X_b)$  is homologically trivial, and so, by Proposition 2.8 is nilpotent.

*Step 4.* We can now finish the proof. Observe that a specialization argument reduces the proof to showing it for a general  $b \in B$ . (cf. [49, Thm. 1.7] and [50, Thm. 0.6]). For general  $b$  the fibre  $X_b$  will be in general position with respect to  $\mathcal{Y}$  and  $\mathcal{X}$  so that

$$\Gamma_{\text{left}} := \Pi_{\text{left}}|_{X_b \times X_b}$$

will be supported on  $Y_b \times X_b$  with  $Y_b$  of dimension  $c$ , and likewise

$$(2) \quad \Gamma_{\text{mid}} := \Pi_{\text{mid}}|_{X_b \times X_b}$$

will factor with a shift  $c$ . Let  $\Gamma_{\text{right}}$  be the transpose of  $\Gamma_{\text{left}}$ . For some  $N \gg 0$  we have

$$(3) \quad \left( \Delta_{X_b} - \Gamma_{\text{left}} - \Gamma_{\text{mid}} - \Gamma_{\text{right}} \right)^{\circ N} = 0 \text{ in } \text{Corr}_0(X_b \times X_b),$$

where  $\Gamma_{\text{left}}, \Gamma_{\text{right}}$  is supported on  $Y_b \times X_b$ , resp. on  $X_b \times Y_b$ , and  $\Gamma_{\text{mid}}$  factors through  $Z_b$  with shift  $c$  as in Eqn. (2).

Since  $\Gamma_{\text{left}}$  is supported on  $Y_b \times X_b$ , Lemma 2.2 implies that its action on  $A_k(X_b)$  is trivial for  $k < \text{codim } Y = d - e$ . The correspondence  $\Gamma_{\text{mid}}$  by construction factors through  $Z_b$  with shift  $c$  and so – by the same Lemma – its action on  $A_k(X_b)$  is trivial, since  $k < c$ . Now expand the expression (3) to conclude that

$$(\Delta_{X_b})_* = (\text{polynomial in } \Gamma_{\text{right}})_* : A_k(X_b) \rightarrow A_k(X_b).$$

Since  $\Delta_{X_b}$  acts as the identity on  $A_k(X_b)$  this implies indeed that  $A_k(X_b)$  is supported on  $Y_b$ , a variety of dimension  $e$ .  $\square$

*Remark 5.3.* It is possible to be more precise: in the situation of Theorem 5.2, we even have that

$$\cdot L^{d-e} : A^{e-k}(X_b) \rightarrow A^{d-k}(X_b)$$

is surjective in the range  $k < \min\{d - e, c\}$ , so the  $k$ -cycles of  $X_b$  are supported on a dimension  $e$  complete intersection. To obtain this, we remark that the  $\Gamma_{\text{right}}$  in the above proof can be expressed in terms of  $L^{d-e}$ , just as in the proof of [28].

Recall that for curves  $A_0^{\text{AJ}} = 0$  and so, if  $A_0(X_b)$  is supported on a curve, we have  $A_0^{\text{AJ}}(X_b) = 0$ . We thus deduce that for  $c = 1, e = 1$  we get the following special case:

**Corollary 5.4.** *Let  $M$  be a smooth  $(d+1)$ -dimensional projective variety for which  $B(M)$  holds and whose (Chow) motive is finite-dimensional. Let  $X_b$ ,  $b \in B$  be the family of all smooth hypersurfaces in a very ample linear system and suppose that*

$$H_d(X_b)^{\text{var}} \neq 0$$

and

$$H_k(X_b) = \widehat{N}^1 H_k(X_b), \quad k = 2, \dots, d$$

for the general  $b \in B$ . Then

$$A_0^{\text{AJ}}(X_b) = 0$$

for all  $b \in B$ .

*Remark 5.5.* (1) In view of Cor. 4.9(1), for  $n = 2$  the condition on the coniveau becomes  $N^1 H_2(X_b) = H_2(X_b)$ , i.e. all cohomology is algebraic. For  $n = 3$  we should have in addition that  $N^1 H_3(X_b) = H_3(X_b)$  that is  $h^{3,0}(X_b) = 0$  as well as the generalized Hodge conjecture for  $H^3(X_b)$ .

(2) Note that in corollary 5.4, there is no condition on  $H_{d+1}(M)$ , so  $p_g(M)$  could be non-zero. In this case, nothing is known about the Chow groups of  $M$ , so it is remarkable that one can at least control the image

$$\text{Im}\left(A_1(M) \rightarrow A_0(X_b)\right).$$

We next come to our second main theorem. It asserts that a "short" niveau filtration on the variable cohomology already has strong implications for the Abel-Jacobi kernels.

**Theorem 5.6.** *Let  $i : X \hookrightarrow M$  be a complete intersection of dimension  $d$ . Suppose that*

- (1)  $B(M)$  holds;
- (2) The Chow motive of  $M$  is finite dimensional;
- (3)  $H_d^{\text{var}}(X) \neq 0$  and for some positive integer  $c$  we have  $H_d^{\text{var}}(X) \subset \widehat{N}^c H_d(X)$ .

Then for  $k < c$  or for  $k > d - c$  we have

$$i^* : A_{k+r}^{\text{AJ}}(M) \rightarrow A_k^{\text{AJ}}(X), \quad i_* : A_k^{\text{AJ}}(X) \hookrightarrow A_k^{\text{AJ}}(M).$$

Moreover, in this range

$$A_k^{\text{var}}(X) = \ker(A_k(X) \xrightarrow{i_*} A_k(M)) = 0,$$

If in addition

- (a)  $H_k(M) = N^{\lfloor \frac{k}{2} \rfloor} H_k(M)$  for  $k \leq d$ , then  $A_k^{\text{AJ}}(X) = 0$  if  $k < c$  or  $k > d - c$ ;
- (b)  $H_k(M) = N^{\lfloor \frac{k+1}{2} \rfloor} H_k(M)$  for  $k \leq d$ , then  $A_k^{\text{hom}}(X) = 0$  if  $k < c$  or  $k > d - c$ .

*Proof.* Let  $X$  be a smooth complete intersection. In Section 2.5 we showed that there is a decomposition

$$\Delta_X = \pi^{\text{fix}}(X) + \pi^{\text{var}}(X)$$

which in cohomology induce projection onto fixed and variable cohomology respectively. By Proposition A.2 there exists relative codimension- $d$  cycles  $\Pi'$  and  $\Pi^{\text{var}}$  on  $\mathcal{X} \times_B \mathcal{X}$  such that  $\Pi'$  comes from  $M \times M \times B$  and  $\Pi^{\text{var}}$  induces  $\pi^{\text{var}}(X)$ . Moreover, the restriction of

$$R = \Delta_{\mathcal{X}/B} - \Pi' - \Pi_d^{\text{var}}$$

to the general fiber is homologically trivial. By Proposition 5.1 there exists a codimension- $d$  cycle  $\gamma$  on  $M \times M$  such that

$$R|_{X \times X} - \gamma|_{X \times X}$$

is rationally equivalent to zero for  $b \in B$  general. In particular  $\gamma|_{X \times X}$  is homologically trivial. Hence  $\gamma|_{X \times X}$  is nilpotent by Proposition 2.8. Let  $N$  be the index of nilpotency of  $\gamma|_{X \times X}$ . We obtain

$$0 = \gamma^{\circ N}|_{X \times X} = (\Delta_X - \pi^{\text{fix}}(X) - \pi^{\text{var}}(X))^{\circ N}.$$

By assumption (3) and Corollary 4.6 the correspondence  $\pi^{\text{var}}(X)$  factors through a correspondence of degree  $-c$  over a variety of dimension  $d - 2c$  and so acts trivially on  $A_k^{\text{AJ}}(X)$  if  $k < c$  or  $k > d - c$ . Setting  $\psi = \pi^{\text{fix}}(X)$ , we find that for some polynomial  $P$  we have  $P(\psi)_* \circ \psi_* = \psi_* \circ P(\psi)_* = \text{id}$  on the Chow groups  $A_k(X)$  with  $k$  in this range and the first assertion follows. For the second, observe that  $\psi$  acts as zero on  $A_k^{\text{var}}(X)$ .

The assumption (a) in the last clause implies that  $\pi^{\text{fix}}(X)$  factors through a curve and so this summand acts trivially on  $A_k^{\text{AJ}}(X)$  for all  $k$ . So then the above argument indeed gives that  $A_k^{\text{AJ}}(X) = 0$  if  $k < c$  or  $k > d - c$ . In case (b),  $\pi^{\text{fix}}(X)$  factors through a point and we obtain  $A_k^{\text{hom}}(X) = 0$  if  $k < c$  or  $k > d - c$ .  $\square$

**Corollary 5.7.** *In the above situation, suppose that  $c = \lfloor \frac{d}{2} \rfloor$ . Then the motive  $h(X)$  is finite-dimensional. Moreover, if for  $M$  we have  $A_k^{\text{AJ}}(M) = 0$  for all  $k$ , then also  $A_k^{\text{AJ}}(X) = 0$  for all  $k$ .*

*Proof.* The assumptions imply surjectivity of  $i^* : A_k^{\text{AJ}}(M, \text{id}, r) \rightarrow A_k^{\text{AJ}}(h(X), \text{id}, 0)$  in the range  $k = 0, \dots, \lfloor \frac{d-2}{2} \rfloor$ . We then apply Vial's result [44], stated in the Appendix as Theorem B.7.  $\square$

## 6. VARIANTS WITH GROUP ACTIONS

Let  $M$  be a projective manifold of dimension  $d + r$  and let  $L_1, \dots, L_r$  be ample line bundles on  $M$  and, as before, set

$$E := L_1 \otimes \cdots \otimes L_r.$$

We assume that a finite group  $G$  acts on  $M$  and on the  $L_j$  and that the linear systems  $|L_j|^G$ ,  $j = 1, \dots, r$  are base point free. The complete intersection in  $M$  corresponding to  $s = (s_1, \dots, s_r) \in \mathbf{P}(H^0(M, E))$  is denoted  $X_s$ . We consider smooth complete intersections coming from  $G$ -invariant hypersurfaces and set accordingly

$$B := \{b \in \mathbf{P}(H^0(M, E)^G) \mid X_b \text{ is smooth}\}.$$

This is Zariski open in  $\mathbf{P}(H^0(M, E)^G)$ .

The graph of the action of  $g \in G$  on  $M$  will be written  $\Gamma_g \subset M \times M$ . As before, we let  $\widetilde{M \times M}$  be the blow up of  $M \times M$  in the diagonal and  $M^{[2]}$  the Hilbert scheme of length 2 subschemes of  $M$  with the natural quotient morphism

$$\mu : \widetilde{M \times M} \rightarrow M^{[2]}.$$

Consider the "bad" locus

$$B_{E,\mu} = \{y \in \widetilde{M \times M} \mid \text{no } s \in H^0(M, E)^G \text{ separates the points} \\ \text{of the length-two scheme } \mu(y)\}.$$

Note that the  $G$ -invariant sections of  $E$  do not separate points in  $G$ -orbits. We demand instead that they separate entire  $G$ -orbits; in fact we want something less stringent, as expressed by the following notion, involving the proper transforms  $\widetilde{\Gamma}_g$  of  $\Gamma_g$  in  $\widetilde{M \times M}$ .

**Definition 6.1.** Assume  $(M, E)$  and  $G$  as above. We say that  $H^0(M, E)^G$  *almost separates* orbits if the "bad" locus  $B_{E,\mu}$  is contained in  $\bigcup_{g \neq \text{id}} \widetilde{\Gamma}_g \cup R_G$ , where  $R_G$  is a (possibly empty) union of components of codimension  $> \dim M = d + r$ .

This demand ensures that  $I_2(E) \rightarrow \widetilde{M \times M}$  is a repeated blow up of a projective bundle so that its cohomology can be controlled. In order to have an analogue of Proposition 5.1, we demand that for  $g \in G$  the endomorphisms

$$\gamma_g^{\text{var}} = [\Gamma_{g,b}]_*^{\text{var}} \in \text{End } H_d(X_b)^{\text{var}}$$

should be independent. This can be tested using the following result.

**Lemma 6.2.** *Let  $\rho : G \rightarrow GL(V)$  be a representation of a finite group on a finite dimensional  $\mathbf{Q}$ -vector space  $V$ . Then the endomorphisms  $\{\rho_g, g \in G\}$  are independent in  $\text{End } V$  if  $G$  is abelian and every irreducible representation occurs in  $V$ .*

*Proof.* This is a consequence of elementary representation theory. We may work over  $\mathbf{C}$ . In the abelian case the group ring  $\mathbf{C}[G]$  is isomorphic to the regular representation of  $G$  and since the former has for its base the irreducible non-isomorphic characters, the elements  $g, g \in G$  give a basis for  $\mathbf{C}[G]$ . The representation  $\rho$  induces an algebra homomorphism  $\tilde{\rho} : \mathbf{C}[G] \rightarrow \text{End } V$  which is injective if every irreducible representation occurs in  $V$ . So the images  $\tilde{\rho}_g, g \in G$  form an independent set.  $\square$

Let us next introduce some notation. Suppose that  $\chi : G \rightarrow \mathbf{Q}$  is a  $\mathbf{Q}$ -character defining an irreducible  $\mathbf{Q}$ -representation  $V_\chi$ , i.e.  $\chi(g) = \text{Tr}(g)|_{V_\chi}$  for all  $g \in G$ . The corresponding projector in the group ring of  $G$  is

$$\pi_\chi = \frac{1}{|G|} \sum_{g \in G} \chi(g)g \in \mathbf{Q}[G]$$

leading to

$$(4) \quad \Gamma_\chi := \frac{1}{|G|} \sum_{g \in G} \chi(g)\Gamma_{g,b} \in \text{Corr}_0(X_b, X_b)$$

acting on the Chow group of  $M$  and on the homology groups of  $M$  as well as the homology of the complete intersections  $X_b$ . The latter action preserves the decomposition into variable and fixed homology. The  $j$ -th Chow group of the motive  $(X, \Gamma_\chi)$  is by definition

$$A_j(X, \Gamma_\chi) = \text{Im}(\Gamma_\chi : A_j(X) \rightarrow A_j(X)) = A_j(X)^\chi,$$

where for any  $G$ -module  $V$  we set

$$V^\chi := \{v \in V \mid g(v) = \chi(g)v \text{ for all } g \in G\} = \{v \in V \mid (\Gamma_\chi)_*v = v\}.$$

Thus  $\Gamma_\chi$  act as the identity on  $V^\chi$ .

We are now ready to formulate a variant of Proposition 5.1. Its validity is shown in the course of the proof of [50, Theorem 3.3].

**Proposition 6.3.** *Let  $(M, E)$ ,  $G$  and  $B \subset \mathbf{P}(H^0(M, E)^G)$  be as above. Suppose that*

- (1)  $H^0(M, E)^G$  almost separates orbits;
- (2) the endomorphisms  $\gamma_g^{\text{var}} \in \text{End } H_d(X_b)^{\text{var}}$ ,  $g \in G$  are linearly independent;
- (3) for general  $b \in B$  one has  $H_d(X_b)^{\text{var}} \neq 0$ .

Then for any  $\mathcal{D} \in A^d(\mathcal{X} \times_B \mathcal{X})^\chi$  with the property that

$$\mathcal{D}|_{X_b \times X_b} = 0 \text{ in } H_{2d}(X_b \times X_b)^\chi,$$

there exists a codimension- $d$  cycle  $\gamma$  on  $M \times M$  such that

$$\mathcal{D}|_{X_b \times X_b} - \gamma|_{X_b \times X_b} = 0 \text{ in } A_d(X_b \times X_b)^\chi$$

for all  $b \in B$ .

Using this variant, the arguments we employed in Section 5 for  $\Delta_X$  can thus be applied to  $\Gamma_\chi$  provided we restrict to  $H_*(X_b)^\chi$ . Since  $\Gamma_\chi$  acts as the identity on  $A_j(X)^\chi$ , the same conclusions as before can be drawn for these Chow groups and we obtain the following results.

**Theorem 6.4.** *Let  $(M, E)$ ,  $G$  and  $B \subset \mathbf{P}(H^0(M, E)^G)$  be as above. Moreover, let  $\chi$  be a  $\mathbf{Q}$ -character for  $G$  and  $\Gamma_\chi$  the associated projector (4). Suppose that*

- (1)  $B(M)$  holds;
- (2)  $H^0(M, E)^G$  almost separates orbits;
- (3) the endomorphisms  $\gamma_g^{\text{var}} \in \text{End } H_d(X_b)^{\text{var}}$ ,  $g \in G$  are linearly independent;
- (4) the Chow motive  $(M, \Gamma_\chi)$  is finite-dimensional.

Assume, moreover, that for a general  $b \in B$  one has  $H_d(X_b)^{\text{var}} \neq 0$  and that

$$H_k(X_b)^\chi \subset \widehat{N}^c H_k(X_b) \text{ for all } k \in \{e+1, \dots, d\}.$$

Then for any  $b \in B$

$$\text{Niveau}((A_j(X_b))^\chi) \leq e - j \text{ for all } j < \min\{d - e, c\},$$

i.e., there exists a subvariety  $Z_b \subset X_b$  of dimension  $d$  such that  $A_j(Z_b) \rightarrow A_j(X_b, \Gamma_\chi)$  is surjective if  $j < \min\{d - e, c\}$ .

**Theorem 6.5.** *Notation as in the previous theorem. Let  $X \subset M$  be a  $G$ -invariant complete intersection of dimension  $d$ . Suppose that*

- (1)  $B(M)$  holds;
- (2)  $H^0(M, E)^G$  almost separates orbits;
- (3) the endomorphisms  $\gamma_g^{\text{var}}$ ,  $g \in G$  are linearly independent in  $\text{End}(H_d(X)^{\text{var}})$ ;
- (4) the Chow motive  $(M, \Gamma_\chi)$  is finite-dimensional;
- (5)  $0 \neq H_n(X)^{\text{var}}$  and for some positive integer  $c$  we have  $H_d(X)^{\text{var}, \chi} \subset \hat{N}^c H^d(X)$ .

Then for  $k < c$  or for  $k > d - c$  we have

$$i^* : A_{k+r}^{\text{AJ}}(M)^\chi \twoheadrightarrow A_k^{\text{AJ}}(X)^\chi, \quad i_* : A_k^{\text{AJ}}(X)^\chi \hookrightarrow A_k^{\text{AJ}}(M)^\chi.$$

Moreover, in this range

$$A_k^{\text{var}}(X)^\chi = \ker(A_k(X)^\chi \xrightarrow{i_*} A_k(M)^\chi) = 0,$$

If in addition  $H_k(M)^\chi = N^{\lfloor \frac{k}{2} \rfloor} H_k(M)^\chi$  for  $k \leq d$ , then  $A_k^{\text{AJ}}(X)^\chi = 0$  if  $k < c$  or  $k > d - c$ .

We also have the analogue of Corollary 5.7:

**Corollary 6.6.** *In the above situation, suppose that  $c = \lfloor \frac{d}{2} \rfloor$ . Then the motive  $h(X, \Gamma_\chi)$  is finite-dimensional.*

## 7. EXAMPLES

**7.1. A threefold of general type with finite dimensional motive.** In [39] one of the authors investigated a quasi-smooth threefold  $X$  which is a complete intersection of three degree 6 hypersurfaces in the weighted projective space  $P = \mathbf{P}(2^4, 3^3)$  and showed that  $A_0(X) = \mathbf{Q}$ . Let us check that this example can also be treated within the present framework. The only technical obstacle is that  $P$  and  $X$  have (mild) singularities, but – as in loc. cit., close inspection of the proofs shows that this does not matter.

The threefold  $X$  is of general type and has Hodge numbers  $h^{1,0}(X) = h^{2,0}(X) = 0$ ,  $h^{1,1} = 1$ ,  $h^{3,0} = 0$ ,  $h^{2,0} = 6$ . Moreover, the intermediate jacobian  $J^2(X)$  is an abelian variety and there is a curve  $C$  and a correspondence  $\gamma \in \text{Corr}_1(C, X)$  inducing a surjection  $J(C) \twoheadrightarrow J^2(X)$ . Hence  $H^3(X) = N^1 H^3(X)$ . Since  $H^2(X) = N^1 H^2(X)$  and  $H^1(X) = 0$  we can apply Cor 5.7 to conclude that  $h(\hat{X})$  is finite dimensional where  $\hat{X}$  is a toroidal resolution of  $X$ . Moreover, the cycle class map is injective in *all* degrees.

**7.2. Hypersurfaces of abelian threefolds.** We let  $A$  be an abelian variety of dimension three. Let  $\iota = -1_A$  be the standard involution. Choose an irreducible principal polarization  $L$  that is preserved by  $\iota$ . The following facts are well known (see e.g. [24]).

*Facts.*

- $L$  is ample and sections of  $L^{\otimes 2}$  correspond to even theta functions (and hence are invariant under the involution).
- $L^3 = 3! = 6$  and  $\dim H^0(L^{\otimes 2}) = 8$ .
- The linear system  $|L^{\otimes 2}|$  defines a 2-to-1 morphism  $\kappa : A \rightarrow \text{Km}(A) \subset \mathbf{P}^7 = \mathbf{P}H^0(L^{\otimes 2})^*$ , where  $\text{Km}(A)$  is the Kummer threefold associated to  $A$ , an algebraic threefold, smooth outside the images of the  $2^6$  two-torsion points of  $A$ .

We let  $X = \{\theta_0 = 0\} \subset A$  be a general divisor in  $|L^{\otimes 2}|$ . This is a smooth surface invariant under  $\iota$  and  $\kappa$  induces an étale double cover of surfaces  $X \rightarrow Y = X/(\iota|_X) \subset \text{Km}(A)$ . The crucial properties of  $A$  are as follows. We use the standard notation for the character spaces for the action of  $\mathbf{Z}/2\mathbf{Z} = \{\text{id}, \iota\}$  on a vector space  $V$ :

$$V^\pm = \{v \in V \mid \iota(v) = \pm v\}.$$

**Proposition 7.1.** (1) We have  $H_1(X)^+ = 0$ ;

(2) the splitting

$$H_2^{\text{var}}(X) = H_2^{\text{var},+}(X) \oplus H_2^{\text{var},-}(X)$$

is non-trivial and  $H_2^{\text{var},+}(X) = N^1 H_2^{\text{var},+}(X)$ , i.e.,  $H^{2,0}(X)^{\text{var},+} = 0$ .

Before giving the proof, we observe that Theorem 6.5 and Corollary 6.6 imply:

**Corollary 7.2.** We have  $A_0^{\text{var}}(X)^+ = 0$  and the motive  $h(X)^+ = h(Y)$  is finite-dimensional (of abelian type).

We now give the

*Proof of Proposition 7.1.* (1) Since  $\iota$  acts as  $-\text{id}$  on one-forms,  $b_1(Y) = b_1(X)^+ = 0$ .

(2) We consider cohomology instead of homology. Consider the Poincaré residue sequence

$$0 \rightarrow \Omega_A^3 \rightarrow \Omega_A^3(X) \xrightarrow{\text{res}} \Omega_X^2 \rightarrow 0.$$

In cohomology this gives

$$0 \rightarrow H^0(\Omega_A^3) \rightarrow H^0(\Omega_A^3(X)) \xrightarrow{\text{res}} H^0(\Omega_X^2) \rightarrow H^1(\Omega_A^3) \rightarrow 0.$$

Since  $H^0(\Omega_A^3(X)) = H^0(L^{\otimes 2})$  we deduce that

$$h_{\text{var}}^{0,2}(X) = 7, \quad h_{\text{fix}}^{0,2}(X) = 3.$$

By the residue sequence, variable holomorphic 2-forms are the Poincaré-residues along  $X$  of meromorphic 3-forms on  $A$  with at most a simple pole along  $X = \{\theta_0 = 0\}$  are given by expressions of the form

$$\frac{\theta}{\theta_0} dz_1 \wedge dz_2 \wedge dz_3$$

with  $\theta$  a theta-function on  $A$  corresponding to a section of  $L^{\otimes 2}$ , and where  $z_1, z_2, z_3$  are holomorphic coordinates on  $\mathbf{C}^3$ . It follows that such forms are *anti-invariant* under  $\iota$  and so  $h_{\text{var}}^{2,0}(X) = h_{\text{var},-}^{2,0}(X) = 7$

To complete the proof, we need to show that  $H_{\text{var},+}^{1,1}(X) = H_{\text{var}}^{1,1}(Y)$  is non-trivial. This is a consequence of the following calculation.  $\square$

**Lemma 7.3.** The invariants of  $X$  and  $Y$  are as follows.

variety	$b_1$	$b_2^{\text{var}} = (h_{\text{var}}^{2,0}, h_{\text{var}}^{1,1}, h_{\text{var}}^{0,2})$	$b_2^{\text{fix}} = (h_{\text{fix}}^{2,0}, h_{\text{fix}}^{1,1}, h_{\text{fix}}^{0,2})$
$X$	6	43 = (7, 29, 7)	15 = (3, 9, 3)
$Y$	0	7 = (0, 7, 0)	15 = (3, 9, 3)



*Proof.* By Lefschetz' theorem  $b_1(X) = b_1(A) = 6$ . To calculate  $b_2$  we observe that  $c_1(X) = -2L|_X$  and  $c_2(X) = 4L^2|_X$  so that

$$c_1^2(X) = c_2(X) = 4L^2|_X = 8L^3 = 48.$$

Since  $c_2(X) = e(X) = 2 - 2b_1(X) + b_2(X) = 48$ , it follows that  $b_2(X) = 58$ . Now  $b_2^{\text{fix},+}(X) = b_2(A) = 15$  and so  $b_2^{\text{var}}(X) = 43$ . The 2-forms on  $X$  that are the restrictions of holomorphic 2-forms on  $A$  are clearly invariant and  $h_{\text{fix}}^{2,0}(X) = h_{\text{fix},+}^{2,0}(X) = 3$ . Since  $h_{\text{var}}^{2,0} = 7$ , the invariants for  $X$  follow.

For  $b_2(Y)$  we use that  $\iota|_X$  acts freely on the generic  $X$  and so  $e(Y) = \frac{1}{2}e(X) = \frac{1}{2}c_2(X) = 24 = 2 + b_2(Y)$  implying that  $b_2(Y) = 22$ . Using Künneth, we find  $b_2^{\text{fix},+}(X) = b_2^+(A) = b_2(A) = 15$  and so  $b_2^{\text{fix},+}(X) = 15$ , and  $b_2^{\text{var},+}(X) = 7$ . Since  $h_{\text{var},+}^{2,0}(X) = 0$ , this yields the invariants for  $Y$ .  $\square$

**7.3. Burniat-Inoue surfaces.** The preceding example can be used to investigate the motive of the classical Burniat-Inoue surfaces. By definition a Burniat surface is a minimal surface  $Y$  of general type with invariants

$$p_g(Y) = q(Y) = 0, \quad e(Y) = 6 \implies b_1(Y) = 0, b_2(Y) = h^{1,1}(Y) = 4.$$

Such surfaces have been constructed by Burniat in [9], while Inoue in [15] gave a different construction as a quotient of a hypersurface in a product of three elliptic curves. It is this construction that we follow.

It has recently been shown by Pedrini-Weibel [37, Theorem 9.1] and, independently, by Bauer-Frapporti [6] that for such  $Y$  one has  $A_0(Y) = \mathbf{Q}$ .<sup>4</sup> We give a different proof fitting our set-up. The reader will notice that our proof is much simpler. To explain the construction of the surface from [15], consider the abelian threefold

$$A := E_1 \times E_2 \times E_3, \quad E_\alpha = \mathbf{C}/\Lambda_\alpha, \quad \text{with } \Lambda_\alpha = \mathbf{Z} \oplus \mathbf{Z}\tau_\alpha, \quad \alpha = 1, 2, 3.$$

and the group  $G$  generated by three commuting involutions

$$\begin{aligned} \iota_1 &: (z_1, z_2, z_3) \mapsto (z_1, -z_2 + \frac{1}{2}, z_3 + \frac{1}{2}) \\ \iota_2 &: (z_1, z_2, z_3) \mapsto (z_1 + \frac{1}{2}, z_2, -z_3 + \frac{1}{2}) \\ \iota_3 &: (z_1, z_2, z_3) \mapsto (-z_1 + \frac{1}{2}, z_2 + \frac{1}{2}, z_3). \end{aligned}$$

We recall some classical facts about theta functions on an elliptic curve  $E$  with period lattice generated by 1 and  $\tau \in \mathfrak{h}$ . The 2-dimensional space  $H^0(E, 2 \cdot [0])$  is generated by two theta-functions. This space is a representation for the group  $G_E$  defined as the group generated by  $\iota$  and the translation  $t_{\frac{1}{2}}$  over the half period  $[\frac{1}{2}]$ . All of  $H^0(E, 2 \cdot [0])$  is invariant under  $\iota$ . One can find two theta functions that are interchanged under  $t_{\frac{1}{2}}$  and we let  $\Theta_E^+, \Theta_E^-$  be their sum, respectively their difference. Then  $H^0(E, 2 \cdot [0]) = (++) \oplus (+-)$  as a  $G_E$ -module. Now set

$$\Theta_{j_1 j_2 j_3} := \Theta_{E_1}^{j_1} \Theta_{E_2}^{j_2} \Theta_{E_3}^{j_3}.$$

<sup>4</sup>In loc. cit. this is in fact shown for the so-called generalized Burniat-type surfaces with  $p_g = q = 0$ .

These give a basis for the space of sections of the line bundle  $\mathcal{O}_{E_1}(2 \cdot [0]) \boxtimes \mathcal{O}_{E_2}(2 \cdot [0]) \boxtimes \mathcal{O}_{E_3}(2 \cdot [0])$  consisting of common eigenvectors for the action of  $G$ . Indeed,  $\mathbf{C}\Theta_{j_1 j_2 j_3} = (j_2 j_3 \ j_1 j_3 \ j_1 j_2)$ .

For generic  $c \in \mathbf{C}$  the equation  $\Theta_{+++} + c\Theta_{--} = 0$  defines a  $G$ -invariant surface  $X$  in  $A$  on which  $G$  acts freely. The quotient  $X = Y/G$  is a classical Burniat-Inoue surface. The crucial observation is that the involution  $j = \iota_1 \iota_2 \iota_3$  is just the standard involution  $x \mapsto -x$  on  $A$ . Then Corollary 7.2 shows that the Chow motive of the surface  $Y/j$  is finite dimensional. This then is also true for  $X = Y/G$ , but since  $p_g(X) = 0$  it follows that automatically  $A_0(X) = \mathbf{Q}$ .

It is worthwhile to note that our argument cannot be applied directly to the group  $G$  since the condition that the endomorphisms  $\gamma_g, g \in G$  be independent, is not fulfilled in this case. See the table below which gives the character spaces.

space	<b>1</b>	<b>-1</b>	<b>- + +</b>	<b>+ - +</b>	<b>+ + -</b>	<b>+ - -</b>	<b>- + -</b>	<b>- - +</b>
$H^{0,1}(A) = H^{0,1}(X)$						1	1	1
$H^{0,2}(A) = H_{\text{fix}}^{0,2}(X)$						1	1	1
$H^{1,1}(A) = H_{\text{fix}}^{1,1}(X)$	3					2	2	2
$H_{\text{var}}^{0,2}(X)$		1	2	2	2			
$H_{\text{var}}^{1,1}(X)$	1							

*Remark 7.4.* A variant of this argument applies to all generalized Inoue-Burniat surfaces, i.e. those surfaces forming the families  $\mathcal{S}_1, \dots, \mathcal{S}_{16}$  from [5]. This will be treated in a forthcoming publication.

**7.4. Hypersurfaces in products of a hyperelliptic curve and a K3-surface.** Let  $C$  be a hyperelliptic curve with hyperelliptic involution  $\iota_C$ , and let  $S$  be a K3-surface with  $h(S)$  finite dimensional and which admits a fixed point free involution  $\iota_2$ . Such surfaces exist, see e.g. the examples of Enriques surfaces in [4, §4] coming from a K3-surface with Picard number  $\geq 19$ . By remark 2.7 the motive of  $S$  – and hence of  $M := C \times S$  – is finite dimensional. The involution  $\iota = (\iota_1, \iota_2)$  acts without fixed points on  $M$ . We let  $L_1$  be the hyperelliptic divisor on  $C$  and we pick a very ample divisor  $L_2$  on  $S$  invariant under the Enriques involution  $\iota_2$  and we set  $L = L_1 \boxtimes L_2$ . Let

$$i : X \hookrightarrow M = C \times S$$

be a smooth hypersurface in  $|L|$  invariant under  $\iota$ . Since  $\iota$  has no fixed points,  $Y = X/\iota$  is a smooth surface. The analogues of Proposition 7.1 and its corollary are valid here.

**Proposition 7.5.** *We have*

- (1)  $H_1(X)^+ = 0$ ;
- (2)  $H^{2,0}(X)^{\text{fix},+} = 0$ ;
- (3) *the splitting*

$$H_2^{\text{var}}(X) = H_2^{\text{var},+}(X) \oplus H_2^{\text{var},-}(X)$$

*is non-trivial and  $H^{2,0}(X)^{\text{var},+} = 0$ ;*

- (4)  $A_0(X)^{\text{var},+} = 0$  and the motive  $h(X)^+ = h(Y)$  is finite-dimensional of abelian type.

*Proof.* To simplify notation, we write

$$2u = L_2^2, \quad u \in \mathbf{Z}$$

which is possible since  $L_2^2$  is even.

*Step 1. Calculation of the Betti numbers of  $X$  and  $Y$ .*

We claim:

- $b_1(X) = 2g$  and  $b_2(X) = 4g + 4(g + 2)u + 46$ ,
- $b_1(Y) = 0$  and  $b_2(Y) = g + (2g + 2)u + 22$ .

To show this, observe that the Künneth formula and the Lefschetz hyperplane theorem imply  $b_1(X) = b_1(M) = b_1(C) = 2g$  and  $b_1(Y) = b_1^+(X) = b_1^+(C) = 0$ . To calculate  $b_2(X)$  we calculate the Euler number  $e(X) = c_2(X)$  from the Whitney product formula

$$(1 + c_1(j^*L))(1 + c_1(X) + c_2(X)) = 1 + (2 - 2g)P_1 + 24P_2 + \dots, \quad P_1 = i^*p_1^*[C], \quad P_2 = i^*p_2^*[S]$$

which gives  $c_1(X) = (2 - 2g)P_1 - c_1(i^*L)$  and hence

$$\begin{aligned} c_2(X) &= 24P_2 - c_1(j^*L)c_1(X) \\ &= 24P_2 + (2g - 2)P_1 \cdot c_1(i^*L) + c_1^2(i^*L) \\ &= 24P_2 + (2g - 2)P_1 \cdot (2P_1 + \ell_2) + (2P_1 + \ell_2)^2, \quad \ell_2 = c_1(i^*p_2^*L_2). \end{aligned}$$

Identifying  $H^4(X, \mathbf{Z})$  with the integers, we have

$$P_1^2 = 0, \quad P_2 = 2, \quad (P_1 \cdot \ell_2) = L_2^2 = 2u, \quad \ell_2^2 = 4u.$$

and so

$$c_2(X) = 48 + 4(g + 2)u = 2 - 4g + b_2(X) \implies b_2(X) = 46 + 4g + 4(g + 2)u.$$

We calculate  $b_2(Y)$  from the Euler number of  $Y$  as follows.

$$\begin{aligned} 2 + b_2(Y) &= e(Y) = \frac{1}{2}e(X) = \frac{1}{2}(2 - 2g + b_2(X)) \\ \implies b_2(Y) &= \frac{1}{2}b_2(X) - (g + 1) = 22 + g + 2(g + 2)u. \end{aligned}$$

*Step 2. Variable and fixed homology.*

Remarking that the fixed cohomology equals  $\text{Im}(i^* : H^2(M) \hookrightarrow H^2(X))$ , we find  $b_2^{\text{fix}}(X) = b_2(M) = b_2(C) + b_2(S) = 23$ . Since  $M/\iota = \mathbf{P}^1 \times \{\text{Enriques surface}\}$ , we find  $b_2^{\text{fix}}(X) = b_2(M/\iota) = 11$ . We put the result in a table.

variety	$b_2^{\text{var}}$	$b_2^{\text{fix}}$
$X$	$4g + 4(g + 2)u + 23$	23
$Y$	$g + (2g + 2)u + 22$	11

*Step 3. Hodge numbers of  $X$ .*

As one readily verifies, the fixed cohomology has Hodge numbers

$$h_{\text{fix}}^{2,0}(X) = 1, \quad h_{\text{fix}}^{1,1}(X) = 21.$$

For the variable cohomology we have

$$h_{\text{var}}^{2,0}(X) = (g + 1) \cdot u + g + 2, \quad h_{\text{var}}^{1,1}(X) = 2(g + 3)u + 2g - 2.$$

To see this consider the Poincaré residue sequence in this situation.

$$0 \rightarrow \Omega_M^3 \rightarrow \Omega_M^3(X) \xrightarrow{\text{res}} \Omega_X^2 \rightarrow 0.$$

From the long exact sequence in cohomology we deduce that

$$(5) \quad \Omega_M^3(X) = p_1^* \omega_C(L_1) \wedge p_2^* \omega_S(L_2) \implies h_{\text{var}}^{2,0}(X) = h^0(C, \omega_C \otimes L_1) \cdot (h^0(S, L_2) - g).$$

By Riemann-Roch  $h^0(C, \omega_C \otimes L_1) = h^1(L_1^*) = g + 1$  and  $h^0(S, L_2) = u + 2$ . The result for  $h_{\text{var}}^{2,0}(X)$  follows.

*Step 4. Hodge numbers of  $Y$ .*

From the fact that  $M/\iota$  is the product of  $\mathbf{P}^1$  and an Enriques surface, we that find  $h_{\text{fix}^+}^{2,0} = 0$  and  $h_{\text{fix}^+}^{1,1} = 11$ . To find the Hodge numbers for the variable cohomology, we use a basic observation.

**Lemma 7.6.** *We have  $h^0(C, \omega_C \otimes L_1)^+ = 0$ .*

*Proof.* Invariant meromorphic 1-forms on  $C$  having a pole at most in the hyperelliptic divisor correspond to meromorphic 1-forms on  $\mathbf{P}^1$  with at most 1 pole. But there are no such forms.  $\square$

As a corollary, from (5) it then follows that  $h_{\text{var}}^{0,2}(X)^+ = 0$  and so  $H^2(X)_{\text{var}}^+$  is pure of type  $(1, 1)$ . We claim that  $H^2(X)_{\text{var}}^+ \neq 0$ . Indeed, our calculations lead to the following table.

variety	$(h_{\text{var}}^{2,0}, h_{\text{var}}^{1,1}, h_{\text{var}}^{0,2})$	$(h_{\text{fix}}^{2,0}, h_{\text{fix}}^{1,1}, h_{\text{fix}}^{0,2})$
$X$	$((g+1)u + g + 2, 2(g+3)u + 2g + 21, (g+1)u + g + 2)$	$(1, 21, 1)$
$Y$	$(0, 2(g+3)u + 2g + 10, 0)$	$(0, 11, 0)$

$\square$

**7.5. Hypersurfaces in products of three curves.** Let  $M = C_1 \times C_2 \times C_3$  where  $C_\alpha$  are curves equipped with an involution  $\iota_\alpha$ . Assume that  $L_\alpha$  is a very ample line bundle on  $C_\alpha$  which is preserved by  $\iota_\alpha$  and such that the system  $|L_\alpha|^{\iota_\alpha}$  gives a morphism. Put  $\iota = (\iota_1, \iota_2, \iota_3)$  and let  $X \subset M$  be a general member of the system  $|L_1 \otimes L_2 \otimes L_3|^\iota$  where we identify  $L_\alpha$  with its pull back to  $M$ . The group  $G$  generated by the three involutions  $\iota_\alpha$  acts on  $M$ . As in the previous subsections, one can calculate the various character spaces for the action of  $G$  on  $H_2(X)^{\text{var}}$ . Suppose one factor, say  $C_1$ , is hyperelliptic. Using Lemma 7.6, one sees that this makes the niveau of  $H_2(X)^{\iota_1, \text{var}}$  equal to 1. Choosing the other factors suitably so that all character spaces appear in  $H_2(X)^{\text{var}}$  one finds (many) projectors  $\pi$  with  $A_0^{\text{AJ, var}}(X, \Gamma_\pi) = 0$ . Let us give one concrete example.

We let  $C_1$  be a genus  $g$  hyperelliptic curve, and  $C_2, C_3$  genus 3 unramified double covers of some genus 2 curve. We take for  $L_1$  the degree 2 hyperelliptic bundle and we take for  $L_\alpha$ ,  $\alpha = 2, 3$  the degree 2 bundles for which the system  $|L_\alpha|$  induces the unramified double cover of  $C_\alpha$  onto the genus 2 curve. Note that  $\iota$  acts without fixed points in this case. As before, we let  $Y = X/\iota$ . We find the following invariants.

variety	$b_1$	$(h_{\text{var}}^{2,0}, h_{\text{var}}^{1,1}, h_{\text{var}}^{0,2})$	$(h_{\text{fix}}^{2,0}, h_{\text{fix}}^{1,1}, h_{\text{fix}}^{0,2})$
$X$	$2(g+6)$	$(7g+16, 14g+477g+16)$	$(6g+9, 12g+21, 6g+9)$
$Y$	8	$(0, 12g+28, 0)$	$(4, 8, 4)$

Concluding,  $H_{\text{var},+}^2(X)$  is pure of type  $(1, 1)$  and  $H_{\text{var}}^2(X)$  contains an invariant and anti-invariant part so that we can apply our considerations to the motive  $(X, \frac{1}{2}(1 + \iota))$  and hence

$$A_0^{\text{AJ},\text{var}}(X)^+ = 0.$$

It follows, as before, that  $h(Y) = h(X)^+$  is finite-dimensional.

*Remark 7.7.* Using [27] we have that the map

$$A_1^{\text{hom}}(Y) \otimes A_1^{\text{hom}}(Y) \rightarrow A_0^{\text{AJ}}(Y)$$

induced by intersection product is surjective, like in the case of an Abelian variety of dimension 2. To see this, consider the commutative diagram

$$\begin{array}{ccc} H^{1,0}(Y) \otimes H^{1,0}(Y) & \xrightarrow{\quad \wedge \quad} & H^{2,0}(Y) \\ \downarrow \simeq & & \downarrow \simeq \\ \otimes^2 (H^{1,0}(C_2/\iota_2) \oplus H^{1,0}(C_3/\iota_3)) & \longrightarrow & H^{1,0}(C_2/\iota_2) \otimes H^{1,0}(C_3/\iota_3), \end{array}$$

which shows that the top-line is a surjection.

**7.6. Odd-dimensional complete intersections of four quadrics.** The following example is due to Bardelli [3]. Let  $\iota : \mathbf{P}^7 \rightarrow \mathbf{P}^7$  be the involution defined by

$$\iota(x_0 : \dots : x_3 : y_0 : \dots : y_3) = (x_0 : \dots : x_3 : -y_0 : \dots : -y_3).$$

Let  $X = V(Q_0, \dots, Q_3)$  be the intersection of four  $\iota$ -invariant quadrics. Then  $H^{3,0}(X)^- = 0$ , hence  $H^3(X)^-$  is a Hodge structure of level one. Bardelli showed that there exist a smooth curve  $C$  and a correspondence  $\gamma \in \text{Corr}_1(C, X)$  such that  $\gamma_* : H_1(C) \rightarrow H_3(X)^-$  is surjective. Hence  $H_3(X)^- \subseteq \tilde{N}^1 H_3(X) = \hat{N}^1 H_3(X)$ . By Theorem 6.5 we get  $A_0^{\text{AJ}}(X)^- = 0$ .

Consider the projector  $p = \frac{1}{2}(\text{id}_X - \iota^*)$ . As  $\iota_* = \iota^*$ , we have  ${}^t p = p$ . Hence the motive  $N = (X, p)$  satisfies  $N \cong N^\vee(3)$  and we can apply Theorem B.7 to the map  $i^* : M = (\mathbf{P}^7, \frac{1}{2}(\text{id}_{\mathbf{P}^7} - \iota^*)) \rightarrow N$ . This shows that the motive  $N = h(X)^-$  is finite dimensional; more precisely, it is a direct factor of  $M^i = M \oplus M^\vee(3) \oplus_i h(C_i)(i)$  for some curves  $C_i$ . As  $A_i^{\text{AJ}}(M^i) = 0$  for all  $i$ , we obtain that

$$A_i^{\text{AJ}}(X)^- = 0$$

for all  $i$ . In other words the quotient morphism  $f : X \rightarrow Y := X/\iota$  induces an isomorphism

$$f^* : A_{\text{AJ}}^*(Y) \xrightarrow{\cong} A_{\text{AJ}}^*(X).$$

This example can be generalised to higher dimension.

**Theorem 7.8.** *Let  $\iota$  be the involution on  $\mathbf{P}^{2m+3}$  ( $m \geq 2$ ) defined by*

$$\iota(x_0 : \dots : x_{m+1} : y_0 : \dots : y_{m+1}) = (x_0 : \dots : x_{m+1} : -y_0 : \dots : -y_{m+1})$$

*and let  $X = V(Q_0, \dots, Q_3)$  be a complete intersection of four  $\iota$ -invariant quadrics. Let  $G = \{\text{id}, \iota\}$  and let  $\chi : G \rightarrow \{\pm 1\}$  be the character defined by  $\chi(\iota) = (-1)^{m-1}$ . Then  $H^{2m-1}(X)^\chi$  is a Hodge structure of level one, and there exist a smooth curve  $C$  and a correspondence  $\gamma \in \text{Corr}_{m-1}(C, X)$  such that  $\gamma_* : H_1(C) \rightarrow H_{2m-1}(X)^\chi$  is surjective.*

*Proof.* See [34, Chapter 3] or [35, Chapter 4].  $\square$

**Corollary 7.9.** *The motive  $h(X)^\times$  is finite dimensional and  $A_i^{\text{AJ}}(X)^\times = 0$  for all  $i$ .*

*Remark 7.10.* The same reasoning can be applied to the examples in [48].

## APPENDIX A. A VARIANT OF VOISIN'S ARGUMENTS

**Proposition A.1.** *Let  $\Gamma$  be a codimension- $k$  cycle on  $\mathcal{X} \times_B \mathcal{X}$  and suppose that for  $b \in B$  very general,*

$$\Gamma|_{X_b \times X_b} \text{ in } H^{2k}(X_b \times X_b)$$

*is supported on  $V_b \times W_b$ , with  $V_b, W_b \subset X_b$  closed of codimension  $c_1$  resp.  $c_2$ . Then there exist closed  $\mathcal{V}, \mathcal{W} \subset \mathcal{X}$  of codimension  $c_1$  resp.  $c_2$ , and a codimension- $k$  cycle  $\Gamma'$  on  $\mathcal{X} \times_B \mathcal{X}$  supported on  $\mathcal{V} \times_B \mathcal{W}$  and such that*

$$\Gamma'|_{X_b \times X_b} = \Gamma|_{X_b \times X_b} \text{ in } H^{2k}(X_b \times X_b)$$

*for all  $b \in B$ .*

*Proof.* Use the same Hilbert schemes argument as in [49, Proposition 3.7], which is the case  $V_b = W_b$ .  $\square$

**Proposition A.2.** *Suppose that  $H_k(X_b) = \widehat{N}^c H_k(X_b)$  for all  $k \in \{e+1, \dots, d\}$  and all  $b \in B$ . Then there exist families  $\mathcal{X}_k \rightarrow B$  of relative dimension  $k - 2c$  and relative degree zero correspondences  $\Pi'_k \in \text{Corr}_B(\mathcal{X}, \mathcal{X})$  such that*

- (a)  $\Pi'_k$  factors through  $\mathcal{X}_k$ ;
- (b)  $\Pi'_k|_{X_b \times X_b}$  is homologous to the  $k$ -th Künneth projector  $\pi_k(X_b)$  for  $k = e+1, \dots, d$ .

*Proof.* Using the assumptions and a Hilbert scheme argument as in [50] there exist a Zariski open subset  $U \subset B$ , a finite étale covering  $\pi : V \rightarrow U$ , a family  $\mathcal{X}_k \rightarrow V$  of relative dimension  $k - 2c$  and relative correspondences  $\Gamma \in \text{Corr}_V(\mathcal{X}_k, \mathcal{X})$ ,  $\Gamma' \in \text{Corr}_V(\mathcal{X}, \mathcal{X}_k)$  such that

$$(*) \quad Q(\Gamma_v(x), y) = Q'(x, \Gamma'_v(y))$$

for all  $x \in H_k(X_{\pi(v)})$ ,  $y \in H_{k-2c}(Z_{\pi(v)})$  and  $v \in V$ . We now consider  $\Gamma$  and  $\Gamma'$  as relative cycles over  $U$ . Let  $u \in U$ . If  $\pi^{-1}(u) = \{v_1, \dots, v_N\}$  we have  $\Gamma_u = \sum_j \Gamma_{v_j}$ ,  $\Gamma'_u = \sum_j \Gamma'_{v_j}$ . As condition  $(*)$  holds for all  $v_j$ , we obtain

$$(*) \quad Q(\Gamma_u(x), y) = Q'(x, \Gamma'_u(y)).$$

We can extend  $\mathcal{X}$  to  $B$  by relative projective completion and desingularisation, and extend  $\Gamma$  and  $\Gamma'$  to relative correspondences over  $B$  by taking their Zariski closure.

As before, let  $H_k^{\text{fix}}(X_b)$  be the image of the restriction map  $H_{k+2r}(M) \rightarrow H_k(X_b)$ . As  $B(M)$  holds there exists an algebraic cycle  $\beta_{d+r-k}$  that induces the operator  $\Lambda^{d+r-k}$ . Set  $R_k = \beta_{d+r-k} \circ L^{d-k} \circ \pi_{k+2r}(M)$ . If we pull back these cycles to  $M \times M \times B$  and then to  $\mathcal{X} \times_B \mathcal{X}$ , we obtain relative correspondences  $\Pi_k \in \text{Corr}_B(\mathcal{X}, \mathcal{X})$  such that  $\Pi_k|_{H_k^{\text{fix}}(X_b)}$  is the identity for all  $k$  (see e.g. [26, Lemmas 3.2 and 3.3]). Note that by construction  $R_k$  factors through a subvariety of dimension  $r+k$  of  $M$  and  $\Pi_k|_{X_b \times X_b}$  factors through a subvariety  $Y_b \subset X_b$  of dimension  $k$ , i.e.,  $\Pi_k|_{X_b \times X_b} \in \text{Im } A_d(Y_b \times X_b) \rightarrow A_d(X_b \times X_b)$ .

Write  $\mathcal{T} = \Gamma \circ \Gamma' \in \text{Corr}_B(\mathcal{X}, \mathcal{X})$ . Replacing  $\mathcal{T}$  by  $\Pi_k \circ \mathcal{T}$  if necessary, we may assume that  $\mathcal{T}|_{X_b \times X_b}$  acts as zero on  $H_j(X_b)$  for all  $j \neq k$ . By construction  $(\mathcal{T}_b)_* : H_k(X_b) \rightarrow H_k(X_b)$  is an isomorphism, hence it has an algebraic inverse by the Cayley-Hamilton theorem as we saw in the proof of Proposition 4.3. We want to perform a relative version of this construction. To this end, note that since  $f : \mathcal{X} \rightarrow B$  is a smooth morphism, the sheaf  $R_k f_* \mathbf{Q}$  is locally constant. Hence there exists an open covering  $\{U_\alpha\}$  of  $B$  and isomorphisms  $f_\alpha$  from  $R_k f_* \mathbf{Q}|_{U_\alpha}$  to the constant sheaf with fiber  $H_k(X_0)$  ( $0 \in U_\alpha$  a base point). As  $\mathcal{T}$  is a relative correspondence defined over  $B$ , the maps  $(\mathcal{T}|_{U_\alpha})_* : R_k f_* \mathbf{Q}|_{U_\alpha} \rightarrow R_k f_* \mathbf{Q}|_{U_\alpha}$  induce automorphisms

$$T_\alpha : H_k(X_0) \rightarrow H_k(X_0)$$

that commute with the transition functions  $f_{\alpha\beta} = f_\alpha \circ f_\beta^{-1}$ :

$$T_\alpha = f_{\alpha\beta} \circ T_\beta \circ f_{\alpha\beta}^{-1}.$$

Hence the characteristic polynomial of  $T_\alpha$  does not depend on  $\alpha$ . This implies that there exists a polynomial  $P(\lambda)$  such that

$$P(\mathcal{T}_b)_* = (\mathcal{T}_b)_*^{-1}$$

for all  $b \in B$ . Define  $\mathcal{U} = P(\mathcal{T}) \in \text{Corr}_B(\mathcal{X}, \mathcal{X})$  and set  $\Pi'_k = \mathcal{U} \circ \mathcal{T}$ .  $\square$

**Corollary A.3.** *There exists relative correspondences  $\Pi_{\text{left}}$ ,  $\Pi_{\text{mid}}$  and  $\Pi_{\text{right}}$  and families  $\mathcal{Y} \rightarrow B$  of relative dimension  $d$ ,  $\mathcal{X} \rightarrow B$  of relative dimension  $d - 2c$  such that*

- (1)  $\Pi_{\text{left}}$  is supported on  $\mathcal{Y} \times_B \mathcal{X}$  and  $\Pi_{\text{right}}$  is supported on  $\mathcal{X} \times \mathcal{Y}$ ;
- (2)  $\Pi_{\text{mid}}$  factors through  $\mathcal{X}$ ;
- (3) The restriction of

$$\Delta_{\mathcal{X}/B} - \Pi_{\text{left}} - \Pi_{\text{mid}} - \Pi_{\text{right}}$$

to  $X_b \times X_b$  is homologous to zero for all  $b \in B$ .

*Proof.* Define  $\Pi_{\text{left}} = \sum_{k=0}^e \Pi_k$ ,  $\Pi_{\text{mid}} = \Pi'_d + \sum_{k=e+1}^{d-1} (\Pi'_k + {}^t \Pi'_k)$  and  $\Pi_{\text{right}} = {}^t \Pi_\ell$ . For the support condition on  $\Pi_\ell$  and  $\Pi_r$  use Proposition A.1.  $\square$

## APPENDIX B. ON A RESULT OF VIAL

In this appendix we give a quick proof of a result of Vial [44] using the work of Kahn–Sujatha [20] on birational motives. We work with the category of covariant motives  $\text{Mot}_{\text{rat}}(k)$ . The Lefschetz object in this category is  $\mathbf{L} = (\text{Spec}(k), \text{id}, 1)$ . The category  $\text{Mot}_{\text{rat}}^0(k)$  of birational motives is the pseudo-abelian completion of the quotient  $\text{Mot}_{\text{rat}}(k)/\mathcal{L}$ , where  $\mathcal{L}$  is the ideal of morphisms that factor through an object of the form  $M \otimes \mathbf{L}$  with  $M \in \text{Mot}_{\text{rat}}(k)^{\text{eff}}$ . We denote the image of a motive  $M$  under the functor

$$\text{Mot}_{\text{rat}}(k) \rightarrow \text{Mot}_{\text{rat}}^0(k)$$

by  $M^0$ . Kahn–Sujatha prove that

$$\text{Hom}_{\text{Mot}^0}(h(X)^0, h(Y)^0) \cong A_0(Y_{k(X)}) \otimes \mathbf{Q}.$$

More generally we have [41]

$$\text{Hom}_{\text{Mot}^0}(h(X)^0, M^0) \cong A_0(M_{k(X)}) \otimes \mathbf{Q}.$$

We shall also use the category  $\text{Mot}_{\text{num}}(k)$  of numerical motives, which is abelian and semisimple [17]. The image of  $M \in \text{Mot}_{\text{rat}}(k)$  under the functor  $\text{Mot}_{\text{rat}}(k) \rightarrow \text{Mot}_{\text{num}}(k)$  is denoted  $\overline{M}$ .

**Lemma B.1.** *Let  $f : M \rightarrow N$  be a morphism in  $\text{Mot}_{\text{rat}}(k)$  such that  $M$  is finite dimensional. If  $\overline{f} : \overline{M} \rightarrow \overline{N}$  admits a left inverse then  $f$  admits a left inverse.*

*Proof.* If  $\overline{g} \circ \overline{f} = \text{id}_{\overline{M}}$  then  $g \circ f - \text{id}_M$  is nilpotent. Writing out the expression  $(g \circ f - \text{id}_M)^N = 0$  we obtain a left inverse for  $f$ .  $\square$

**Lemma B.2.** *Let  $f : M \rightarrow N$  be a morphism in  $\text{Mot}_{\text{rat}}(k)$ . If  $M$  is finite dimensional, there exists a decomposition  $N \cong N_1 \oplus N_2$  such that*

- (1)  $N_1$  is isomorphic to a direct factor  $M_1$  of  $M$  (hence finite dimensional);
- (2)  $\overline{N}_1 \cong \text{Im } \overline{f}$ .
- (3) The composition  $M \rightarrow N \rightarrow N_2$  is numerically trivial.

*Proof.* In  $\text{Mot}_{\text{num}}(k)$  we have decompositions  $\overline{M} \cong \overline{M}_1 \oplus \overline{M}_2$  and  $\overline{N} \cong \overline{N}_1 \oplus \overline{N}_2$  such that  $\overline{M}_1 \rightarrow \overline{N}_1$  is an isomorphism and the remaining maps  $\overline{M}_i \rightarrow \overline{N}_j$  are zero. Since  $M$  is finite dimensional, the direct summand  $\overline{M}_1$  lifts to a direct summand  $M_1$  of  $M$ . Put  $\alpha = f|_{M_1} : M_1 \rightarrow N$ . As  $\overline{\alpha}$  is a monomorphism it admits a left inverse. By Lemma B.1 there exists  $\beta : N \rightarrow M_1$  such that  $\beta \circ \alpha = \text{id}_{M_1}$ . Define  $\pi = \alpha \circ \beta \in \text{End}(N_1)$ . Then  $\pi$  is a projector and we have  $N = N_1 \oplus N_2$  with  $N_1 = (N, \pi)$  and  $N_2 = (N, \text{id} - \pi)$ . Then  $M_1 \cong N_1$  and by construction  $\overline{N}_1 \cong \text{Im } \overline{f}$  and  $\overline{M} \rightarrow \overline{N}_2$  is the zero map.  $\square$

**Lemma B.3.** *Let  $M = (X, p, m)$  and  $N = (Y, q)$ , and let  $f : M \rightarrow N$  be a morphism in  $\text{Mot}_{\text{rat}}(k)$  such that*

- (1)  $M$  is finite dimensional;
- (2)  $A_0(M_\Omega) \rightarrow A_0(N_\Omega)$  is surjective, with  $\Omega \supset k$  a universal domain;
- (3)  $f$  is numerically trivial.

*Then  $N^0 = 0$  in  $\text{Mot}_{\text{rat}}^0(k)$ .*

*Proof.* The second assumption implies that

$$\begin{array}{ccc} A_0(M_{k(Y)}) & \rightarrow & A_0(N_{k(Y)}) \\ \parallel & & \parallel \\ \text{Hom}(h(Y)^0, M) & \rightarrow & \text{Hom}(h(Y)^0, N) \end{array}$$

is surjective, hence there exists  $\varphi \in \text{Hom}(h(Y)^0, M^0)$  such that  $f^0 \circ \varphi = q^0 = \text{id}_{N^0}$ . In particular,  $f^0 : M^0 \rightarrow N^0$  is an epimorphism. Write  $\varphi = p^0 \circ \psi$  with  $\psi : h(Y)^0 \rightarrow h(X)^0$ . There exists  $\gamma \in \text{Corr}_{-n}(Y, X)$  such that  $\gamma^0 = \psi$ . Put  $g = p \circ \gamma \circ q : N \rightarrow M$  and  $\pi = g \circ f \in \text{End}(M)$ . As  $\overline{f} = 0$ ,  $\overline{\pi} = 0$ . Hence  $\pi$  is nilpotent since  $M$  is finite dimensional. By construction  $f^0 \circ g^0 = \text{id}_{N^0}$ , so  $\pi^0$  is a projector and by nilpotence we get  $\pi^0 = 0$ . This implies that  $f^0 = f^0 \circ \pi^0 = 0$ , hence  $N^0 = 0$  since  $f^0 : M^0 \rightarrow N^0$  is an epimorphism.  $\square$

**Remark B.4.** Suppose  $k = \mathbf{C}$ . It suffices to assume that  $A_0^{\text{AJ}}(M) \rightarrow A_0^{\text{AJ}}(N)$  is surjective. Indeed, there exists a curve  $C$  such that  $J(C) \rightarrow \text{Alb}(N)$  is surjective. We then replace  $M$  by  $M' = M \oplus h(C)$  and apply the Lemma to  $M'$ .



**Corollary B.5.** *Let  $f : M = (X, p, m) \rightarrow N = (Y, q)$  be a morphism in  $\text{Mot}_{\text{rat}}(k)$  such that*

- (1)  *$M$  is finite dimensional;*
- (2)  *$A_i(M_\Omega) \rightarrow A_i(N_\Omega)$  is surjective for all  $i \leq \ell - 1$ .*

*Then  $N \cong N_1 \oplus N_2$  with  $N_1$  finite dimensional and  $N_2 \cong (Z, \rho, \ell)$  with  $\dim Z = d - \ell$ .*

*Proof.* By Lemma B.2  $N \cong N_1 \oplus N_2$  with  $M \rightarrow N_2$  numerically trivial, hence  $N_2^0 = 0$  by Lemma B.3. This implies that  $N_2 \cong R(1)$  with  $R = (Z, \rho)$  and  $\dim Z = d - 1$ . This finishes the proof if  $\ell = 1$ . The general case follows by induction on  $\ell$  using the formula  $A_i(R(k)) = A_{i-k}(R)$ .  $\square$

**Remark B.6.** Assume  $k = \mathbf{C}$ .

- (1) As noted before, it suffices to assume that

$$A_i^{\text{AJ}}(M_\Omega) \rightarrow A_i^{\text{AJ}}(N_\Omega)$$

is surjective for all  $i \leq \ell - 1$  (here  $\Omega = \mathbf{C}$  considered as universal domain).

- (2) If the motive  $M$  is self-dual up to twist, i.e.,  $M \cong M^\vee(d)$ , the statement of the Corollary can be improved. Write  $N = N_1 \oplus R(\ell)$  as before, and consider the map  $M \cong M^\vee(d) \rightarrow R(\ell)^\vee(d) = R^\vee(d - \ell) = (Z, {}^t\rho) = R'$ . By assumption  $A_i(M_\Omega) \rightarrow A_i(R'_\Omega)$  is surjective for all  $i \leq \ell - 1$ , hence  $R' \cong R'_1 \oplus R'_2$  such that  $R'_1$  is finite dimensional and  $R'_2 = (Z', \rho', 2\ell - d)$  with  $\dim(Z') = \dim Z - \ell = d - 2\ell$ .

Summarizing, we get the following result.

**Theorem B.7** (Vial). *Let  $f : M = (X, p, m) \rightarrow N = (Y, q)$  be a morphism in  $\text{Mot}_{\text{rat}}(\mathbf{C})$  such that  $M$  is finite dimensional.*

- (1) *If  $A_i^{\text{AJ}}(M) \rightarrow A_i^{\text{AJ}}(N)$  is surjective for all  $i \leq d - 1$  then  $N$  is isomorphic to a direct factor of  $M \oplus \bigoplus_{i=1}^d h(C_i)(i)$  where  $C_i$  is a smooth curve for all  $i$ .*
- (2) *If  $M \cong M^\vee(d)$  and  $A_i^{\text{AJ}}(M) \rightarrow A_i^{\text{AJ}}(N)$  is surjective for all  $i \leq \frac{d-2}{2}$  then  $N$  is isomorphic to a direct factor of  $M \oplus M^\vee(d) \oplus \bigoplus_i h(C_i)(i)$  with  $C_i$  smooth curves.*

*Hence  $N$  is finite dimensional in both cases.*

*Proof.* Use Corollary B.5 and the previous Remark.  $\square$

**Remark B.8.** The proof of Theorem B.7 gives a bit more: if the motive  $M$  is “of abelian type” (i.e., belongs to the subcategory of  $\text{Mot}_{\text{rat}}(\mathbf{C})$  generated by the motives of abelian varieties over  $k$ ) then  $N$  is of abelian type.

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