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# INFINITE ORBIT DEPTH AND LENGTH OF MELNIKOV FUNCTIONS

PAVAO MARDEŠIĆ, DMITRY NOVIKOV, LAURA ORTIZ-BOBADILLA,  
AND JESSIE PONTIGO-HERRERA

**ABSTRACT.** In this paper we study polynomial Hamiltonian systems  $dF = 0$  in the plane and their small perturbations:  $dF + \epsilon\omega = 0$ . The first nonzero Melnikov function  $M_\mu = M_\mu(F, \gamma, \omega)$  of the Poincaré map along a loop  $\gamma$  of  $dF = 0$  is given by an iterated integral [3]. In [7], we bounded the length of the iterated integral  $M_\mu$  by a geometric number  $k = k(F, \gamma)$  which we call *orbit depth*. We conjectured that the bound is optimal.

Here, we give a simple example of a Hamiltonian system  $F$  and its orbit  $\gamma$  having infinite orbit depth. If our conjecture is true, for this example there should exist deformations  $dF + \epsilon\omega$  with arbitrary high length first nonzero Melnikov function  $M_\mu$  along  $\gamma$ . We construct deformations  $dF + \epsilon\omega = 0$  whose first nonzero Melnikov function  $M_\mu$  is of length three and explain the difficulties in constructing deformations having high length first nonzero Melnikov functions  $M_\mu$ .

## 1. INTRODUCTION AND MAIN RESULTS

This paper is motivated by two classical problems in the study of orbits of vector fields in the plane: the 16-th Hilbert problem and the center problem or rather their infinitesimal versions.

The Infinitesimal Hilbert 16-th problem asks for a bound on the number of limit cycles (i.e. isolated periodic orbits) created by a small polynomial deformation of a given degree of an integrable vector field in the plane.

The infinitesimal center problem asks for a characterization of polynomial deformations of an integrable system which preserve a family of loops.

In both problems one studies the Poincaré first return map (1.2) on a transversal. The first (possibly) nonzero term  $M_\mu$  carries lots of information about the Poincaré map. Having an a priori estimate on its complexity would be very important for both infinitesimal problems. For the infinitesimal center problem, to have an a priori estimate on the length is similar to having an estimate on the stabilization index for Noetherian property.

It is known [2, 3], that when deforming a Hamiltonian vector field, the first nonzero term  $M_\mu$  is an iterated integral of length not exceeding its order  $\mu$ . However, the order  $\mu$  in general depends on the deformation. In [7], we gave a bound on the length of  $M_\mu$  by a geometric number *orbit depth*  $k$  which is independent on the deformation. We showed that in different cases this bound is optimal and we conjectured that it is so in general.

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In this paper we give an example where this bound is infinite. We believe that in the example one can construct deformations whose first nonzero Melnikov function  $M_\mu$  is of arbitrarily high length. In that direction we construct for our example deformations having first nonzero Melnikov function  $M_\mu$  of length 3 and show the difficulties in constructing deformations with higher length.

*Remark 1.1.*

- (i) Our example answers negatively a question asked by Gavrilov and Iliev in [4].
- (ii) Our example shows the complexity of both infinitesimal problems.

Let us be more precise. Let  $F \in \mathbb{C}[x, y]$  be a polynomial and let  $\gamma \in \pi_1(F^{-1}(t))$  be a loop for  $t$  a regular value of  $F$ . Consider a small polynomial deformation

$$dF + \epsilon\omega = 0, \quad (1.1)$$

of the Hamiltonian  $dF = 0$ . Let  $\tau$  be a transversal section to  $\gamma$  at a point  $p_0$ , parametrized by the values  $t$  of  $F$ . Denote by  $P_\gamma$  the Poincaré return map (holonomy) of (1.1) along  $\gamma$ . Then

$$P_\gamma(t) = t + \epsilon^\mu M_{\gamma, \mu}(t) + o(\epsilon^\mu). \quad (1.2)$$

If the Poincaré map is not the identity map, we assume that  $M_\mu$  is nonzero and call it the *first non-zero Melnikov function along  $\gamma$  of the deformation (1.1)*.

By the Poincaré-Pontryagin criterion, the first order Melnikov function  $M_1$  is given by an Abelian integral,

$$M_{\gamma, 1}(t) = \int_\gamma \omega. \quad (1.3)$$

More generally,  $M_{\gamma, \mu}(t)$  is given as a linear combination of iterated integrals of length at most  $\mu$ , see [2, 3]. However, this bound in general is not optimal. For instance, for generic  $F$  and any loop  $\gamma$  and any deformation  $\omega$ , the first non-zero Melnikov functions  $M_{\gamma, \mu}(t)$  is given by an Abelian integral (i.e. is an iterated integral of length 1), irrespective of its order  $\mu$ . This follows from [6, 2], see [7]. For other examples see [7], as well as papers cited there. Moreover, the bound  $\mu$ , for the length of  $M_{\gamma, \mu}$  depends on the deformation (1.1).

In [4] a sufficient condition under which the first nonzero Melnikov function  $M_{\gamma, \mu}(t)$  is an Abelian integral is formulated. We generalized this condition in [7]:

Let  $\Sigma$  be the set of atypical values of  $F$ , see [5], and let  $t \notin \Sigma$  be some regular value of  $F$ . Denote  $\Gamma_t = \{F^{-1}(t)\}$ . The fundamental group  $\pi_1(\mathbb{C} \setminus \Sigma, t)$  acts on the fundamental group  $\pi_1(\Gamma_t, p_0)$  as follows. For each generator  $a_j$  of  $\pi_1(\mathbb{C} \setminus \Sigma, t)$  corresponding to a closed curve  $a_j(s) \subset \mathbb{C} \setminus \Sigma$ , choose its lifting  $\tilde{a}_j(s)$ , i.e. a loop  $\tilde{a}_j(s) \subset F^{-1}(\mathbb{C} \setminus \Sigma)$  such that  $F(\tilde{a}_j(s)) = a_j(s)$  and  $\tilde{a}_j(0) = \tilde{a}_j(1) = p_0$ . Then, by Ehresmann's fibration theorem, the fundamental groups  $\pi_1(F^{-1}(a_j(s)), \tilde{a}_j(s))$  and  $\pi_1(F^{-1}(a_j(s')), \tilde{a}_j(s'))$  are canonically isomorphic for sufficiently close  $s, s'$ . This defines an automorphism  $Mon(a_j)$  of  $\pi_1(\Gamma_t, p_0)$  and the representation  $Mon : \pi_1(\mathbb{C} \setminus \Sigma, t) \rightarrow Aut(\pi_1(\Gamma_t, p_0))$ . This representation depends on the choice of the liftings  $\tilde{a}_j$ , and different choices of liftings change  $Mon(a_j)$  to conjugate automorphisms  $\sigma_j^{-1} Mon(a_j) \sigma_j$ ,  $\sigma_j \in \pi_1(\Gamma_t, p_0)$ . We fix some choice of  $\tilde{a}_j$ .

**Definition 1.2** (see [4, 7]). Let  $\mathcal{O}$  be the smallest normal subgroup of  $\pi_1(\Gamma_t, p_0)$  containing the orbit of  $\gamma \in \pi_1(\Gamma_t, p_0)$  under the action of  $Mon(\pi_1(\mathbb{C} \setminus \Sigma, t))$ . Denote  $K = [\mathcal{O}, \pi_1(\Gamma_t, p_0)]$  and let  $H_1(\mathcal{O}) = \mathcal{O}/K$ .

*Remark 1.3.* Note that  $\mathcal{O}$ ,  $K$  and  $H_1(\mathcal{O})$  are independent on the particular choice of  $\tilde{a}_j$ . Moreover,  $H_1(\mathcal{O})$  is canonically isomorphic for different choices of  $p_0$  in the following sense: the natural isomorphism between  $\pi_1(\Gamma_t, p_0)$  and  $\pi_1(\Gamma_t, p'_0)$  defined by a path joining  $p_0$  and  $p'_0$  descends to an isomorphism of the corresponding  $H_1(\mathcal{O})$ , and this isomorphism is independent on the choice of this path.

In what follows, we denote  $\pi_1(\Gamma_t, p_0)$  by  $\pi_1$ . The *lower central sequence* of  $\pi_1$  is defined as:

$$\pi_1 = L_1 \supset L_2 = [L_1, \pi_1] \supset \cdots \supset L_{i+1} = [L_i, \pi_1] \supset \cdots \quad (1.4)$$

There is a natural homomorphism  $\iota_1 : H_1(\mathcal{O}) \rightarrow H_1(\Gamma_t, \mathbb{C})$ , and let  $\mathcal{O}_1 = \iota_1(H_1(\mathcal{O})) = \frac{\mathcal{O}L_2}{L_2} \otimes \mathbb{C}$ . In general,  $\iota_1$  is neither surjective nor injective. In [4] it is shown that if  $\iota_1$  is injective then  $M_\mu(t)$  is an Abelian integral.

In [7], we defined the *orbit depth*  $k = k(F, \gamma)$ ,

**Definition 1.4.** Given a polynomial  $F \in \mathbb{C}[x, y]$  and a loop  $\gamma \in \pi_1$  as above, the orbit depth  $k = k(F, \gamma)$  is defined as

$$k = \sup \{j \geq 1 \mid \mathcal{O} \cap L_j \not\subseteq K\} \subset \mathbb{N} \cup \{+\infty\}. \quad (1.5)$$

We say that an element  $v \in \mathcal{O}$  is of depth  $j$  if it belongs to  $L_j$  and its class in  $H_1(\mathcal{O})$  is nonzero. Orbit depth is  $k < \infty$  if  $k$  is the highest depth of elements in  $\mathcal{O}$ , and it is infinite if there are elements of  $\mathcal{O}$  of arbitrary high depth.

In [7, Theorem 1.7]) we proved that the orbit depth  $k = k(F, \gamma)$  bounds the length of iterated integrals representing the first nonzero Melnikov function  $M_{\gamma, \mu}$  of small deformations (1.1).

We conjectured that it was an optimal bound for the length of the first nonzero Melnikov function  $M_{\gamma, \mu}$  along  $\gamma$  of deformations of  $dF = 0$ . We hence believe that for a Hamiltonian system  $dF = 0$  and a loop  $\gamma \in \pi_1(F^{-1}(t))$  of infinite orbit depth there exist polynomial deformations (1.1) such that the first non-zero Melnikov function  $M_{\gamma, \mu}$  is an iterated integral of arbitrary high length.

**Theorem 1.5.** *There exists a polynomial function  $F \in \mathbb{R}[x, y]$  and a loop  $\gamma \in \pi_1(F^{-1}(t))$  such that the orbit depth  $k$  of  $\gamma$  is infinite.*

*Such an example is given by*

$$F(x, y) = (x^2 - 1)(y^2 - 1), \quad (1.6)$$

*and the loop  $\gamma \subset \{F = t\}$  given by the real cycle vanishing at  $(0, 0)$  along the path  $(0, t) \subset \mathbb{R}$ , for  $t \in (0, 1)$  (see Figure 1).*

Our theorem also answers negatively to the question if  $\dim H_1(\mathcal{O}) \leq \dim H_1(F^{-1}(t_0))$ , which was raised as part of open question (1) in [4].

We also prove

**Theorem 1.6.** *There exist a rational deformation  $dF + \epsilon\omega$  of  $F$  given by (1.6) such that the first nonzero Melnikov function  $M_{\gamma, \mu}$  of the deformation (1.1) is an iterated integral of length 3. An example of such deformation is a form  $\omega$  of type*

$$\omega = a_1(F) \frac{dx}{x+1} + a_2(F) \frac{dy}{y-1} + a_3(F) \frac{dx}{x-1}, \quad (1.7)$$

*with  $a_1(t) = t^2 + 2t$ ,  $a_2(t) = t$  and  $a_3(t) = t^2 + t$ .*

*If  $M_{\gamma, 2} = M_{\gamma, 3} \equiv 0$  for deformation (1.1) with  $\omega$  as (1.7), then the deformation is integrable.*

We conclude that one needs a richer set of deformations to get an example of a perturbation with first nonzero Melnikov function  $M_{\gamma,\mu}$  of length  $\geq 4$ .

One of the principal tools of the proof is Proposition 3.4, establishing connection between Poincaré return maps of paths on  $\Gamma_t$  and the vector fields on the transversal  $\tau$  whose flows give these maps.

2. EXAMPLE WITH INFINITE ORBIT DEPTH

We consider the polynomial  $F(x, y) = (x^2 - 1)(y^2 - 1)$ . The critical values of  $F$  are 0 and 1, and the critical points are  $(\pm 1, \pm 1)$  on  $\{F = 0\}$  and  $(0, 0)$  at  $\{F = 1\}$ . Our goal in this section is to show that the orbit depth of the real cycle  $\gamma$  vanishing at the critical point  $(0, 0)$  is infinity.

The normalizations  $\Gamma_t$  of complexifications of non-singular level curves  $\{F = t\}$ ,  $t \neq 0, 1$ , are torii with 4 points removed. The fundamental group  $\pi_1(\Gamma_t, p_0)$  is a free group generated by loops  $\gamma, \delta_0, \delta_1, \delta_2, \delta_3$ , where  $\delta_i$  are loops vanishing at  $(\pm 1, \pm 1)$ .

To be more precise, we take  $0 < t \ll 1$ , choose  $p_0$  close to the edge  $\{x = -1\}$  of the square, and denote  $\delta_0, \delta_1, \delta_2, \delta_3$  the geometric loops vanishing at  $(-1, -1)$ ,  $(1, -1)$ ,  $(1, 1)$  and  $(-1, 1)$  correspondingly, see the figure at [7]: we take a meridian of the cylinder which is  $\{F = t\}$  near the corresponding singular point, with base point on  $\gamma$ , and then pull the base point clockwise along  $\gamma$  to  $p_0$ . We orient  $\gamma$  counterclockwise, and orient  $\delta_i$  in such a way that the intersection numbers  $(\gamma, \delta_i)$  are all equal to one.

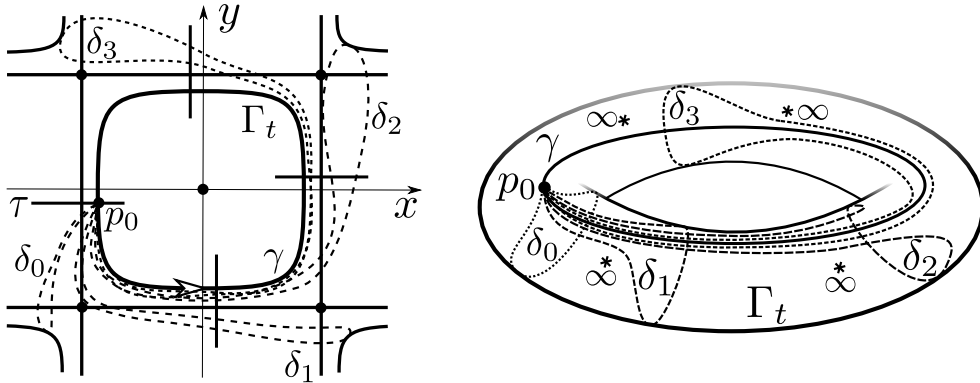


FIGURE 1. Generators of  $\pi_1(\Gamma_t, p_0)$ .

The atypical values of  $H$  are exactly its critical values 0, 1. Therefore, the action of the monodromy of the foliation on the fundamental group of  $\Gamma_t$  is generated by two automorphisms  $Mon_0$  and  $Mon_1$  of  $\pi_1(\Gamma_t, p_0)$  corresponding to the loops going around the critical values 0 and 1 correspondingly, as described above.

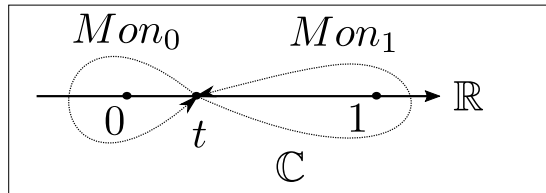


FIGURE 2. Generators of  $\pi_1(\mathbb{C} \setminus \{0, 1\}, t)$  corresponding to  $Mon_0, Mon_1$ .

**Lemma 2.1.** Denote  $\delta = \delta_0\delta_1\delta_2\delta_3$ . The monodromy operators  $Mon_{0,1}$  are

$$Mon_1 = \{\gamma \mapsto \gamma, \delta_i \mapsto \gamma\delta_i\} \quad (2.1)$$

$$\begin{aligned} Mon_0 = \{ & \gamma \mapsto \delta\gamma, \delta_0 \mapsto \delta_0, \\ & \delta_1 \mapsto \delta_0\delta_1\delta_0^{-1}, \\ & \delta_2 \mapsto \delta_0\delta_1\delta_2\delta_1^{-1}\delta_0^{-1}, \\ & \delta_3 \mapsto \delta_0\delta_1\delta_2\delta_3\delta_2^{-1}\delta_1^{-1}\delta_0^{-1}\} \end{aligned} \quad (2.2)$$

*Proof.* These formulas follow from standard homotopical computations proving the Picard-Lefschetz formula, see e.g. [1]. From the local topology in a neighborhood of the center critical point, we have  $Mon_1\gamma = \gamma$ , and  $Mon_1\delta_i = \gamma\delta_i$ , since the intersection number between  $\gamma$  and  $\delta_i$  is one, and  $\gamma$  is the cycle vanishing at the center critical value when the regular value tends to 1. For the monodromy around the critical value 0, we divide the real cycle  $\gamma$  into pieces  $\gamma = \rho_0\rho_1\rho_2\rho_3$ , where  $\rho_i$  goes from a point  $x_i$  in  $\gamma \cap U_i$  to a point  $x_{i+1}$  in  $\gamma \cap U_{i+1}$ , where  $U_j$  is a neighborhood of the critical point at which  $\delta_j$  vanish, with  $x_0$  and  $x_4$  equal to the chosen initial point  $p_0$ . Let  $\delta_i^{\circ}$  be the generator of  $\pi_1(F^{-1}(t) \cap U_i, x_i)$ . From Picard-Lefschetz formula, locally at the neighborhood  $U_i$  we have  $Mon_0(\rho_i) = \delta_i^{\circ}\rho_i$ , and  $Mon_0\delta_i^{\circ} = \delta_i^{\circ}$ . Notice that  $\delta_i = \rho_0 \cdots \rho_i\delta_i\rho_i^{-1} \cdots \rho_0^{-1}$ . Then, applying the local monodromy at each saddle critical point we get the result.  $\square$

**2.1. Chipping out Homology and  $Mon_1$ .** Note that  $\delta = Mon_0(\gamma)\gamma^{-1}$  is in  $\mathcal{O}$ , and that  $\gamma, \delta$  span the orbit of  $\gamma$  in  $H_1(\Gamma_t)$ . From the exact sequence

$$0 \rightarrow (\mathcal{O} \cap L_2)/K \rightarrow \mathcal{O}/K \rightarrow \mathcal{O}_1 \rightarrow 0,$$

where  $\mathcal{O}_1 = \langle \gamma, \delta \rangle \subset H_1(\Gamma_t)$ , it is clear that the next step is to consider the action of the monodromy on  $L_2 = [\pi_1, \pi_1]$ . It turns out that, up to subgroup generated by  $\gamma$  and  $\delta$ , the action of  $Mon_1$  on  $L_2$  is trivial, thus allowing to disregard  $Mon_1$ .

Let  $\Gamma$  be the normal subgroup of  $\pi_1$  generated by  $\gamma, \delta$ . Evidently,  $\Gamma \subset \mathcal{O}$  and  $[\Gamma, \pi_1] \subset K$  is a normal subgroup of  $\pi_1$  generated by commutators  $[\gamma, c], [\delta, c], c \in \pi_1$ .

The group  $(\mathcal{O} \cap L_2)/K$  is a subgroup of  $L_2/K$ , which is a factor of  $L_2/[\Gamma, \pi_1]$ . The latter is isomorphic to the commutator  $G_2 = [G, G]$  of the free group  $G$  generated by  $\delta_1, \delta_2, \delta_3$ .

**Lemma 2.2.**  $Mon_1$  preserves both  $L_2$  and  $\Gamma$ . The induced action of  $Mon_1$  on  $\pi_1/\Gamma$  is trivial.

*Proof.* As  $Mon_1$  is an automorphism of  $\pi_1$ , it preserves  $L_2$ . Also,  $Mon_1(\gamma) = \gamma$ ,  $Mon_1(\delta) = \gamma\delta_0\gamma\delta_1\gamma\delta_2\gamma\delta_3 = \delta \bmod \Gamma \in \Gamma$ , so  $Mon_1$  preserves  $\Gamma$ . Also  $Mon_1(\delta_i) = \gamma\delta_i = \delta_i \bmod \Gamma$ , which proves the last statement.  $\square$

As  $\Gamma \cap L_2 \subset K$ , this implies that  $Mon_1$  acts trivially on  $L_2/K$  and therefore can be disregarded.

**Corollary 2.3.**  $\mathcal{O}$  is generated by  $\gamma$  and  $Mon_0^i(\delta)$ ,  $i = 0, 1, \dots$

**Lemma 2.4.**  $Mon_0$  preserves  $L_i \cap \langle \delta_0, \delta_1, \delta_2, \delta_3 \rangle$  and the induced action of  $Mon_0$  on  $L_i \cap \langle \delta_0, \delta_1, \delta_2, \delta_3 \rangle / (L_{i+1} \cap \langle \delta_0, \delta_1, \delta_2, \delta_3 \rangle)$  is trivial.

*Proof.* Follows immediately from Lemma 2.1.  $\square$

So we have to investigate the orbit of  $\delta$  in the free group generated by  $\langle \delta_i, i = 0, 1, 2, 3 \rangle$ , under the action of  $Mon_0$  given by Lemma 2.1.

Define  $M(\sigma) = \delta_1^{-1}\delta_0^{-1}Mon_0(\sigma)\delta_0\delta_1$  on  $\langle \delta, \delta_1, \delta_2, \delta_3 \rangle$ ,

$$M = \{\delta_0 \mapsto \delta_1^{-1}\delta_0\delta_1, \delta_1 \mapsto \delta_1, \delta_2 \mapsto \delta_2, \delta_3 \mapsto [\delta_2, \delta_3]\delta_3\}, \quad (2.3)$$

and define  $Var(\sigma) = M(\sigma)\sigma^{-1}$ . Note that for  $\sigma \in \mathcal{O}$

$$Var(\sigma) = [\delta_1^{-1}\delta_0^{-1}, Mon_0(\sigma)]Mon_0(\sigma)\sigma^{-1} = Mon_0(\sigma)\sigma^{-1} \bmod K \in \mathcal{O}. \quad (2.4)$$

This means that both  $Mon_0$  and  $M$  generate the same orbit, and one can use either of them. However,  $M$  is computationally more convenient. We formally define  $Var(\gamma) = \delta$  and  $Var^{i+1}(\gamma) = Var^i(\delta)$ .

**Corollary 2.5.** *For any  $i \geq 1$ ,  $Var^i(\gamma) \in L_i \cap \langle \delta, \delta_1, \delta_2, \delta_3 \rangle$ .*

*Proof.* We have  $Var(\gamma) = \delta \in \pi_1 = L_1$ . Therefore  $Var^i(\gamma)$  belong to the subgroup generated by  $\delta_i$ . By induction, and using Lemma 2.4 and (2.4), we see that  $Mon_0(Var^i(\gamma)) = Var^i(\gamma) \bmod L_{i+1} \cap \langle \delta, \delta_1, \delta_2, \delta_3 \rangle$ . Therefore  $Var^{i+1}(\gamma) \in L_{i+1} \cap \langle \delta, \delta_1, \delta_2, \delta_3 \rangle$ .  $\square$

**Lemma 2.6.** *Any element  $w \in \mathcal{O}$  can be represented as*

$$w = \gamma^{n_0} Var(\gamma)^{n_1} \dots Var^i(\gamma)^{n_k} \bmod K, \quad i = i(w). \quad (2.5)$$

*Proof.* Indeed, any element in  $\mathcal{O}/K$  is a product of  $M^i(\gamma)$ , and these elements can be represented in this form: if

$$M^i(\gamma) = \gamma^{n_0} \delta^{n_1} \dots Var^i(\gamma)^{n_k} \bmod K, \quad (2.6)$$

then

$$\begin{aligned} M^{i+1}(\gamma) &= M(\gamma^{n_0})M(\delta^{n_1}) \dots M(Var^i(\gamma)^{n_k}) \bmod K \\ &= (Var(\gamma)\gamma)^{n_0} (Var(\delta)\delta)^{n_1} \dots (Var^{i+1}(\gamma)Var^i(\gamma))^{n_k} \bmod K \\ &= \gamma^{n_0} \delta^{n_0+n_1} \dots Var^i(\gamma)^{n_{k-1}+n_k} Var^{i+1}(\gamma)^{n_k} \bmod K, \end{aligned}$$

as  $Var^i(\gamma)$  commute modulo  $K$ .  $\square$

Define by induction the maps  $d_k : \pi_1 \rightarrow \pi_1$  as  $d_1 = Id$ , and  $d_{k+1}(\sigma) = [\delta_2, d_k(\sigma)]$ . Note that  $d_k(\sigma) = [\delta_2, [\delta_2, [\dots [\delta_2, \sigma] \dots]] \in L_k$  for all  $k \geq 1$ ,  $\sigma \in \pi_1$ .

**Proposition 2.7.** *Denote  $x = \delta_1\delta_2$ ,  $z = \delta_2\delta_3$  and define*

$$v_1 = \delta, \quad v_k = [x, d_{k-1}(z)] \text{ for } k \geq 2. \quad (2.7)$$

*Then  $v_i \in \mathcal{O}$  and  $Var^i(\gamma) = v_i \bmod K$ .*

Before proving Proposition 2.7, we prove

**Lemma 2.8.**  $d_{k-1}([\delta_2, z]z) = d_k(z)d_{k-1}(z)$ .

*Proof.* By induction,

$$\begin{aligned} d_k([\delta_2, z]z) &= [\delta_2, d_{k-1}([\delta_2, z]z)] = [\delta_2, d_k(z)d_{k-1}(z)] = \\ &= [\delta_2, d_k(z)][\delta_2, d_{k-1}(z)] [[d_{k-1}(z), \delta_2], d_k(z)] = \\ &= d_{k+1}(z)d_k(z)[d_k(z)^{-1}, d_k(z)] = d_{k+1}(z)d_k(z). \quad \square \end{aligned}$$

*Proof of Proposition 2.7.* From Lemma 2.1 we see that  $v_1 = Mon_0(\gamma)\gamma^{-1} = \delta$ .

Note that by (2.3)

$$M(x) = x, \quad M(\delta_2) = \delta_2, \quad M(z) = \delta_2 z \delta_2^{-1} = [\delta_2, z]z. \quad (2.8)$$

We have

$$\delta = \delta_1 \delta_2 \delta_3 \delta_0 [\delta_0^{-1}, \delta^{-1}] = \delta_1 \delta_2 \delta_3 \delta_0 \bmod K, \quad (2.9)$$

so, modulo  $K$ ,

$$\begin{aligned} Var^2(\gamma) &= M(\delta_1 \delta_2 \delta_3 \delta_0) \delta^{-1} = \delta_1 \cdot \delta_2 \cdot \delta_2 \delta_3 \delta_2^{-1} \cdot \delta_1^{-1} \delta_0 \delta_1 \cdot \delta^{-1} = \\ &= [\delta_1 \delta_2, \delta_2 \delta_3] [\delta_2 \delta_3, \delta] = [x, z] = v_2. \end{aligned} \quad (2.10)$$

In particular,  $v_2 \in \mathcal{O}$ .

For the third variation, again modulo  $K$ ,

$$\begin{aligned} \text{Var}^3(\gamma) &= M([x, z])v_2^{-1} = [x, [\delta_2, z]z]v_2^{-1} = \\ &= [x, [\delta_2, z]][x, z][z, x], [\delta_2, z]v_2^{-1} = [x, [\delta_2, z]]. \end{aligned} \quad (2.11)$$

as  $[[z, x], [\delta_2, z]] = [v_2^{-1}, [\delta_2, z]] \in K$ .

Now, from (2.8) follows  $M(v_k) = [x, d_{k-1}([\delta_2, z]z)]$ , so modulo  $K$ ,

$$\begin{aligned} \text{Var}^{k+1}(\gamma) &= M(v_k)v_k^{-1} = [x, d_k(z)d_{k-1}(z)]v_k^{-1} = \\ &= [x, d_k(z)][x, d_{k-1}(z)] [[d_{k-1}(z), x], d_k(z)]v_k^{-1} = \\ &= [x, d_k(z)]v_k [v_k^{-1}, d_k(z)]v_k^{-1} = [x, d_k(z)] = v_{k+1}. \quad \square \end{aligned}$$

**Proposition 2.9.**  $\mathcal{O} = \langle \gamma, v_i, i = 1, \dots \rangle$ .

*Proof.* Denote  $\mathcal{O}^v = \langle \gamma, v_i, i = 1, \dots \rangle$  the normal subgroup of  $\pi_1$  generated by  $\gamma, v_i$ . It follows from Proposition 2.7 that  $v_i \in \mathcal{O}$ , so  $\mathcal{O}^v \subset \mathcal{O}$ .

Let us prove the opposite inclusion. By Lemma 2.6,  $\mathcal{O} = \langle \gamma, \text{Var}^i(\gamma), i = 1, \dots \rangle$ . By Lemma 2.7,

$$\text{Var}^i(\gamma) = v_i W_{i,1}(\{[a_j^1, \text{Var}^j(\gamma)]\}_{j \geq 0}), \quad i \geq 0, \quad (2.12)$$

where  $W_{i,1}$  are some words. Substituting these equalities into their right hand sides, we get

$$\text{Var}^i(\gamma) = v_i W_{i,2}(\{[b_j, v_j], [a_j^2, [a_j^1, \text{Var}^j(\gamma)]]\}_{j \geq 0}), \quad i \geq 0. \quad (2.13)$$

Repeating substitution  $\ell - 1$  times, we get for any  $\ell \geq 1$

$$\text{Var}^i(\gamma) = v_i \epsilon_{i,\ell} w_{i,\ell}, \quad \epsilon_{i,\ell} \in \mathcal{O}^v, \quad w_{i,\ell} \in [\pi_1, [\pi_1, [\dots [\pi_1, \mathcal{O}] \dots]] \subset L_{\ell+1}. \quad (2.14)$$

This implies that  $\mathcal{O}L_{\ell+1} \subset \mathcal{O}^v L_{\ell+1}$  for any  $\ell \geq 1$ . This implies that  $\mathcal{O} \subset \mathcal{O}^v$ .  $\square$

**Corollary 2.10.**  $K = \langle [\pi_1, \gamma], [\pi_1, v_i], i = 1, \dots \rangle$ .

**2.2. Depth is infinite.** Here we prove Theorem 1.5. The main idea is to construct for any  $k$  a matrix representation  $\rho_k$  of  $\pi_1$  sending all generators of  $\mathcal{O}$  except  $v_{k+2}$  to identity. We prove that  $\rho_k(v_{k+2}) \notin \rho_k(K)$  for a generic choice of parameters  $a, c$  of the representation  $\rho_k$ , which implies Theorem 1.5.

Let  $A_0 = a$ ,  $B_0 = 1$  and  $C_0 = c$ , where  $a, c \neq 0, 1$ . Define inductively the  $2^k \times 2^k$ -matrices  $A_k, B_k, C_k$  as follows:

$$A_{k+1} = \begin{bmatrix} A_k & 0 \\ 0 & \mathbb{I} \end{bmatrix}, B_{k+1} = \begin{bmatrix} B_k & \mathbb{I} \\ 0 & B_k \end{bmatrix}, C_{k+1} = \begin{bmatrix} \mathbb{I} & 0 \\ 0 & C_k \end{bmatrix}, \quad (2.15)$$

where  $\mathbb{I}$  is the corresponding identity matrix.

**Proposition 2.11.** *Let  $x = \delta_1 \delta_2$  and  $z = \delta_2 \delta_3$  be as in Proposition 2.7. Consider the representation  $\rho_k : \pi_1 \rightarrow GL(2^k)$  defined by  $\rho_k(\gamma) = \rho_k(\delta) = \mathbb{I}$ ,  $\rho_k(x) = A_k$ ,  $\rho_k(\delta_2) = B_k$  and  $\rho_k(z) = C_k$ . Then*

- (1)  $\rho_k(v_i) = \mathbb{I}$  for  $i \neq k+2$  and  $\rho_k(v_{k+2}) \neq \mathbb{I}$ , and
- (2)  $\rho_k(v_{k+2}) \notin [\rho_k(v_{k+2}), \rho_k(\pi_1)] = \rho_k(K)$  for generic  $a, c$ .

*Remark 2.12.* As  $\pi_1$  is a free group and  $\gamma, \delta, x, \delta_2, z$  are its generators, such a representation  $\rho_k$  exists.

*Proof.* We start with another description of  $A_k, B_k$  and  $C_k$ . Let

$$\mathbb{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad J_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Recall that tensor products are multiplied factorwise:

$$(X_1 \otimes Y_1)(X_2 \otimes Y_2) = X_1 X_2 \otimes Y_1 Y_2.$$



Denote by  $b_{j_1 \dots j_l}$  to be the tensor product of  $(k-l)$  copies of  $\mathbb{I}_2$  and  $l$  copies of  $J_2$ , with  $J_2$  being exactly the  $j_1^{\text{th}}, \dots, j_l^{\text{th}}$  factors. Similarly, denote by  $e_{j_1 \dots j_l}$  to be the tensor product of  $(k-l)$  copies of  $E_2$  and  $l$  copies of  $J_2$ , with  $J_2$  being exactly the  $j_1^{\text{th}}, \dots, j_l^{\text{th}}$  factors. Finally, denote  $\alpha = F_2^{\otimes k}$ ,  $\gamma = E_2^{\otimes k}$  and  $\beta = \sum_{j=1}^k b_j$ .

Using this notations, we have

$$A_k = \mathbb{I}_{2^k} + (a-1)\alpha, \quad B_k = \mathbb{I}_{2^k} + \beta \quad \text{and} \quad C_k = \mathbb{I}_{2^k} + (c-1)\gamma. \quad (2.16)$$

Our immediate goal is to compute  $[B_k, C_k]$ . Evidently,

$$A_k^{-1} = \mathbb{I}_{2^k} + \left(\frac{1}{a} - 1\right)\alpha, \quad C_k^{-1} = \mathbb{I}_{2^k} + \left(\frac{1}{c} - 1\right)\gamma. \quad (2.17)$$

As  $J_2^2 = 0$ , we have  $b_{j_1 \dots j_l} b_{j'_1 \dots j'_l}$  equals  $b_{j_1 \dots j_l j'_1 \dots j'_l}$  if the sets  $\{j_1 \dots j_l\}, \{j'_1 \dots j'_l\}$  do not intersect, and zero otherwise.

Therefore

$$\beta^l = l! \sum_{1 \leq j_1 < \dots < j_l \leq k} b_{j_1 \dots j_l}, \quad \beta^k = k! J_2^{\otimes k} \quad \text{and} \quad \beta^{k+1} = 0. \quad (2.18)$$

In particular,

$$B_k^{-1} = \mathbb{I}_{2^k} + \tilde{\beta}, \quad \text{where} \quad \tilde{\beta} = -\beta + \beta^2 - \dots \pm \beta^k.$$

Now, from  $N_2^2 = N_2$ ,  $N_2 J_2 = 0$ ,  $J_2 N_2 = J_2$  we have

$$\gamma^2 = \gamma, \quad \gamma \beta^l = \gamma \tilde{\beta} = 0, \quad (2.19)$$

and

$$\beta^l \gamma = \epsilon^{[l]} = l! \sum_{1 \leq j_1 < \dots < j_l \leq k} e_{j_1 \dots j_l}, \quad (2.20)$$

Note that  $\epsilon^{[l]}\epsilon^{[l']} = 0$  for all  $l, l' \geq 1$ ,  $\epsilon^{[k]} = k! J_2^{\otimes k}$  and  $\epsilon^{[l]} = 0$  for  $l > k$ .

Now, using the above formulae we see that

$$\begin{aligned} [B_k, C_k] &= (\mathbb{I}_{2^k} + \beta)(\mathbb{I}_{2^k} + (c-1)\gamma)(\mathbb{I}_{2^k} + \tilde{\beta})(\mathbb{I}_{2^k} + (\frac{1}{c} - 1)\gamma) \\ &= \mathbb{I}_{2^k} - (\frac{1}{c} - 1)\beta\gamma = \mathbb{I}_{2^k} - (\frac{1}{c} - 1)\epsilon^{[1]}, \end{aligned} \quad (2.21)$$

and, as  $(\epsilon^{[1]})^2 = 0$ ,

$$[B_k, C_k]^{-1} = \mathbb{I}_{2^k} + (\frac{1}{c} - 1)\epsilon^{[1]}. \quad (2.22)$$

Continuing,

$$[B_k, [B_k, C_k]] = \mathbb{I}_{2^k} - (\frac{1}{c} - 1)\beta\epsilon^{[1]} = \mathbb{I}_{2^k} - (\frac{1}{c} - 1)\epsilon^{[2]}, \quad (2.23)$$

and, by induction, for a commutator with  $l$  entries of  $B_k$ ,

$$[B_k, [\dots [B_k, C_k] \dots]] = \mathbb{I}_{2^k} - (\frac{1}{c} - 1)\epsilon^{[l]}. \quad (2.24)$$

Now, similarly, from

$$F_2^2 = F_2, \quad F_2 J_2 = J_2, \quad J_2 F_2 = 0 \quad (2.25)$$

we have

$$\begin{aligned} \alpha^2 &= \alpha, & \epsilon^{[l]}\alpha &= 0 \text{ for all } l, \\ \alpha\epsilon^{[k]} &= \epsilon^{[k]} = k! J_2^{\otimes k}, & \alpha\epsilon^{[l]} &= 0 \text{ for } l \neq k. \end{aligned}$$

Therefore

$$\rho_k(v_{l+2}) = [A_k, [B_k, [\dots [B_k, C_k] \dots]] = \mathbb{I}_{2^k} + (\frac{1}{c} - 1)(\frac{1}{a} - 1)\alpha\epsilon^{[l]}, \quad (2.26)$$

i.e.

$$\begin{aligned} \rho_k(v_{l+2}) &= \mathbb{I}_{2^k} & \text{for } l+2 \neq k, \\ \rho_k(v_{k+2}) &= \mathbb{I}_{2^k} + (\frac{1}{c} - 1)(\frac{1}{a} - 1)k! J_2^{\otimes k} \neq \mathbb{I}_{2^k}, \end{aligned} \quad (2.27)$$

which proves the first claim of Proposition 2.11.

Let  $s = \prod g_i^{m_i} \in \pi_1$ , where  $g_i \in \{\gamma, \delta, x, \delta_2, z\}$  and  $m_i \in \mathbb{Z}$ . Then  $\rho_k(s) = D + U$ , where  $U$  is a strictly upper triangular matrix and  $D = \text{diag}(a^m, 1, \dots, 1, c^n)$ . Therefore

$$[\rho_k(s), \rho_k(v_{k+2})] = \mathbb{I}_{2^k} + \left(\frac{a^m}{c^n} - 1\right) \left(\frac{1}{c} - 1\right) \left(\frac{1}{a} - 1\right) k! J_2^{\otimes k}. \quad (2.28)$$

Now, assume

$$\rho_k(v_{k+2}) = \prod_j [\rho_k(s_j), \rho_k(v_{k+2})]. \quad (2.29)$$

By (2.27), (2.28), we have

$$\mathbb{I}_{2^k} + \left(\frac{1}{c} - 1\right) \left(\frac{1}{a} - 1\right) k! J_2^{\otimes k} = \mathbb{I}_{2^k} + \left(\sum \left(\frac{a^{m_j}}{c^{n_j}} - 1\right)\right) \left(\frac{1}{c} - 1\right) \left(\frac{1}{a} - 1\right) k! J_2^{\otimes k}, \quad (2.30)$$

or, equivalently,  $1 = \sum \left(\frac{a^{m_j}}{c^{n_j}} - 1\right)$ . Collecting similar terms, we get

$$\sum \lambda_i \frac{a^{m_i}}{c^{n_i}} = 1 + \sum \lambda_i, \quad \text{where } \lambda_i, m_i, n_i \in \mathbb{Z},$$

and for any  $i$  one of the exponents  $m_i, n_i$  is non-zero. This cannot hold for all  $a, c$ : if the left hand side is a constant, then all  $\lambda_i$  vanish, and we get  $0 = 1$ . Therefore any representation (2.29) fails on a Zariski open subset of  $\mathbb{C}_{(a,c)}$ , so all such representations fail for a generic choice of  $a, c$ .  $\square$

*Proof of Theorem 1.5.* By Corollary 2.10 and Proposition 2.11(1), we have  $\rho_k(K) = [\rho_k(\pi_1), \rho_k(v_{k+2})]$ . By Proposition 2.11(2),  $\rho_k(v_{k+2}) \notin [\rho_k(\pi_1), \rho_k(v_{k+2})]$  for a generic choice of  $a, c$ . This means that  $(\mathcal{O} \cap L_{k+2}) \setminus K$  contains  $v_{k+2}$ , so is non-empty for all  $k$ .  $\square$

### 3. FIRST NONZERO MELNIKOV FUNCTION OF LENGTH 3

**3.1. Cohomologies: notations.** Denote  $f_1 = x + 1$ ,  $f_2 = y - 1$ ,  $f_3 = x - 1$  and  $f_4 = y + 1$ . Denote  $\phi_i = \log f_i$  and  $\eta_i = d\phi_i = \frac{df_i}{f_i}$ .

The cycles  $\gamma, \delta, \delta_1, \delta_2, \delta_3$  form a basis of  $H_1(\Gamma_t)$ , and  $\gamma, \delta$  form a basis of the orbit of  $\gamma$  in  $H^1(\Gamma_t)$ . As  $\phi_i$  are univalued on  $\gamma$ , the restrictions to  $\Gamma_t$  of polynomial forms  $\{F\eta_i\}_{i=1}^3$  lie in the orthogonal complement  $\mathcal{O}^\perp \subset H^1(\Gamma_t)$  of the orbit  $\mathcal{O} \subset H_1(\Gamma_t)$  of  $\gamma$  in  $H_1(\Gamma_t)$ , and in fact form its basis. We have

$$\begin{aligned} \int_{\delta_1} \eta_1 &= 0, & \int_{\delta_1} \eta_2 &= 0, & \int_{\delta_1} \eta_3 &= 2\pi i \\ \int_{\delta_2} \eta_1 &= 0, & \int_{\delta_2} \eta_2 &= 2\pi i, & \int_{\delta_2} \eta_3 &= -2\pi i \\ \int_{\delta_3} \eta_1 &= 2\pi i, & \int_{\delta_3} \eta_2 &= -2\pi i, & \int_{\delta_3} \eta_3 &= 0. \end{aligned} \quad (3.1)$$

Note that  $d\eta_i = 0$ , so the Gelfand-Leray derivatives  $\frac{d\eta_i}{dF}$  vanish.

**3.2. Linear perturbations.** Consider the rational 1-form of type (1.7), i.e.

$$\omega = a_1(F)\eta_1 + a_2(F)\eta_2 + a_3(F)\eta_3, \quad (3.2)$$

where  $a_i(t)$  are holomorphic on  $\tau$ , and consider the perturbation

$$dF + \epsilon\omega = 0, \quad F = (x^2 - 1)(y^2 - 1). \quad (3.3)$$

*Remark 3.1.* Note that restriction to  $\Gamma_t$  of any form  $\omega$  such that  $\int_{\gamma(t)} \omega \equiv 0$  is cohomologous to a linear combination of  $\eta_i$ .

The Poincaré map along the cycles  $\gamma$  is

$$P_\gamma(t) = t + \epsilon M_{\gamma,1}(t) + \epsilon^2 M_{\gamma,2}(t) + \epsilon^3 M_{\gamma,3}(t) + \dots, \quad (3.4)$$

with  $M_{\gamma,1}(t) = \int_{\gamma(t)} \omega \equiv 0$ . Our goal is to find a polynomial form  $\omega$  providing the highest possible order of the first non-vanishing Melnikov function  $M_{\gamma,i}(t)$

**Proposition 3.2.**  $M_{\gamma,2} \equiv 0$  and  $M_{v_3,3} \neq 0$  if, and only if,

$$\begin{aligned} a_3(t) &= \alpha_1 \int_0^t \frac{\alpha_2(\tau)}{\alpha_1^2(\tau)} d\tau + c_0 \alpha_1 \\ a_1(t) &= a_3(t) + \alpha_1(t) \\ a_2(t) &= \lambda \alpha_1, \end{aligned} \tag{3.5}$$

where  $\lambda \in \mathbb{C}^*$  and  $\alpha_1(t)$  and  $\alpha_2(t)$  are linearly independent functions over  $\mathbb{C}$ , and  $\alpha_1$  is not constant.

To prove Proposition 3.2, we consider the second and third variations of  $\gamma$ , i.e.  $v_2$  and  $v_3$  from Proposition 2.7, and the corresponding Poincaré maps  $P_{v_2}$  and  $P_{v_3}$ . Proposition 3.4 implies that

$$\begin{aligned} P_{v_2}(t) &= t + \epsilon^2 M_{v_2,2}(t) + O(\epsilon^3), \\ P_{v_3}(t) &= t + \epsilon^3 M_{v_3,3}(t) + O(\epsilon^4), \end{aligned} \tag{3.6}$$

and provides explicit expression of  $M_{v_2,2}(t)$ ,  $M_{v_3,3}(t)$  in terms of coefficients  $a_i(F)$ . This allows to find conditions on  $a_i$  guaranteeing  $M_{v_3,3}(t) \neq 0$  and  $M_{v_2,2}(t) \equiv 0$ . We prove that the last condition is equivalent to  $M_{\gamma,2}(t) \equiv 0$  in Lemma 3.7.

*Remark 3.3* (Geometric interpretation of Proposition 3.2). The forms (3.2) form a three dimensional module  $\Omega$  over the ring of germs of holomorphic functions at  $t$ . The Poincaré map along  $\gamma$  is a map  $\mathcal{P}_\gamma : U \subset \Omega \rightarrow \mathcal{H}ol(\tau)$ , where  $\mathcal{H}ol(\tau)$  is the set of germs of holomorphic mappings  $g : (\tau, p_0) \rightarrow \tau$ .

The perturbations (1.1) are germs of lines in  $\Omega$ , and the order of the first non-zero Melnikov function of the perturbation can be interpreted as the order of vanishing of  $\mathcal{P}$  on these lines, i.e. the order of tangency of these lines to the set  $\{\mathcal{R} = 0\}$  of integrable perturbations. Theorem 1.6 claims that the maximum order of this tangency is either at most three or the line lies entirely in  $\{\mathcal{R} = 0\}$ .

To construct the perturbations with first non-zero Melnikov function  $M_k$  of higher length, we necessarily have to increase  $k$ , i.e. the order of tangency of the perturbation with the set of integrable foliations. This means that we have either to consider non-linear perturbations, i.e. germs of curves in  $\Omega$ , or consider a wider class  $\tilde{\Omega}$  of perturbations, e.g. by including relatively exact forms.

Still, the first non-zero Melnikov function of a non-linear perturbation

$$dF + \epsilon \omega_1 + \epsilon^2 \omega_2 + \dots = 0, \quad \omega_i \in \Omega \tag{3.7}$$

can be of high order, but of small length. It is easy to see that the terms of highest length of the corresponding Melnikov functions depend only on  $\omega_1$ . Thus, to ensure that the length of the first non-vanishing Melnikov functions is at least 4, we should take  $\omega_1$  such that  $dF + \epsilon \omega_1 = 0$  is integrable (otherwise  $M_3 \neq 0$ ), and find non-linear terms in such a way that  $M_{\gamma,4} \equiv 0$  (as its longest terms are determined by  $\omega_1$ , they necessarily vanish, so its length could be at most 3), but  $M_{\gamma,5} \neq 0$  and has length 4 (it cannot be of length 5 by the same reason). The latter would follow from  $M_{v_4,5} \neq 0$ . This program can be realized, but it is computationally hard. Moreover, it is not clear how one can generalize this approach to higher length, so we omit the computations.

### 3.3. Poincaré maps as time-one flows of vector fields on the transversal.

Consider the family (1.1) as a one-dimensional foliation

$$\mathcal{F} = \{dF + \epsilon \omega = 0, d\epsilon = 0\} \tag{3.8}$$

in  $\mathbb{C}_{x,y,\epsilon}^3$ . Let  $\Theta = \tau \times (\mathbb{C}_\epsilon, 0)$  be a transversal to the algebraic leaf  $\Gamma = \Gamma_t \times \{0\}$  at the point  $(p_0, 0)$ , and denote  $\mathcal{D} = \text{Diff}((\Theta, (p_0, 0)))$  the group of germs of holomorphic diffeomorphisms of  $\Theta$ . Holonomy of  $\mathcal{F}$  along various paths  $\gamma \in \pi_1(\Gamma, (p_0, 0))$  defines

a representation  $\tilde{P} : \pi_1 \rightarrow \mathcal{D}$  preserving  $\epsilon$ , i.e.  $\tilde{P}_\gamma : (x, \epsilon) \rightarrow (P_\gamma(x, \epsilon), \epsilon)$  for any  $\gamma \in \pi_1$ .

Define  $v_\gamma = (d\tilde{P}_\gamma)(\partial_\epsilon)$ ,  $v_e = \partial_\epsilon$ , and let  $\phi_\gamma^s$  be the  $s$ -time flow of  $v_\gamma$  (necessarily  $L_{v_\gamma}(\epsilon) = 1$ ). By definition,  $\tilde{P}_\gamma$  conjugates flows of  $v_e$  and  $v_\gamma$ . In particular, for all  $p \in \tau$

$$\tilde{P}_\gamma(p, \epsilon) = \tilde{P}_\gamma(\phi_e^\epsilon(p, 0)) = \phi_\gamma^\epsilon(\tilde{P}_\gamma(p, 0)) = \phi_\gamma^\epsilon(p, 0), \quad (3.9)$$

as  $\tilde{P}_\gamma(p, 0) \equiv (p, 0)$  for all  $\gamma \in \pi_1$ .

Let  $(t = F(p), \epsilon)$  be parameterization of  $\Theta$ . The expansion (1.2) is the expansion of  $\tilde{P}_\gamma$  in degrees of  $\epsilon$ ,

$$\tilde{P}_\gamma(t, \epsilon) = (t + \epsilon^\mu M_{\gamma, \mu}(t) + o(\epsilon^\mu), \epsilon). \quad (3.10)$$

Let

$$v_\gamma = (v_\gamma^0(t) + \epsilon v_\gamma^1(t) + \dots) \partial_t + \partial_\epsilon \quad (3.11)$$

be decomposition of  $v_\gamma$ . Evidently,  $v_\gamma^0 = \dots = v_\gamma^{\mu-2} \equiv 0$ , and  $v_\gamma^{\mu-1}(t) = M_{\gamma, \mu}(t)$ .

**Proposition 3.4.** *Let  $\gamma_1, \gamma_2, \gamma = [\gamma_1, \gamma_2] \in \pi_1$ , and let*

$$P_{\gamma_i}(t, \epsilon) = t + \epsilon^{\mu_i} M_{\gamma_i, \mu_i} + o(\epsilon^{\mu_i}), \quad i = 1, 2,$$

with  $M_{\gamma_1, \mu_1}, M_{\gamma_2, \mu_2} \not\equiv 0$ .

Then  $v_\gamma^0 = \dots = v_\gamma^{\mu-2} \equiv 0$  for  $\mu = \mu_1 + \mu_2$ , and

$$v_{[\gamma_1, \gamma_2]}^{\mu-1}(t) = W(M_{\gamma_1, \mu_1}(t), M_{\gamma_2, \mu_2}(t)), \quad (3.12)$$

where  $W(f, g) = fg' - f'g$  denotes the Wronskian of  $f, g$ .

Alternatively,

$$v_{[\gamma_1, \gamma_2]}^{\mu-1}(t) \partial_t = [M_{\gamma_1, \mu_1}(t) \partial_t, M_{\gamma_2, \mu_2}(t) \partial_t], \quad (3.13)$$

where brackets denote the Lie bracket of vector fields.

*Remark 3.5.* Essentially, (1.1) induces a homomorphism  $\mathfrak{R}$  of the fundamental group  $\pi_1$  of  $\Gamma_t$  to the group of germs at identity of analytic curves in the groupoid  $Diff(\tau)$ .

More precisely, for any  $\gamma \in \pi_1$  we get a germ  $\mathfrak{R}(\gamma)$  at identity of an analytic curve

$$\{\hat{\omega}_\gamma\} = \left\{ \tilde{P}_\gamma(\cdot, \epsilon) \right\} \subset Diff(\tau).$$

The Lie algebra  $\mathcal{X}$  of  $Diff(\tau)$  "is" the Lie algebra of germs at  $p_0$  of vector fields on  $\tau$ , and  $v_\gamma$  defines the corresponding (under exponential map) path  $\log \hat{\omega}_\gamma$  in this Lie algebra. The path  $\tilde{P}_\gamma(\cdot, \epsilon)$  is not necessarily a one-parametric group, and the path  $\log \hat{\omega}_\gamma$  is not necessarily constant, but we are interested in the leading term of  $\hat{\omega}_\gamma$  only. If  $\mu = 1$ , then the leading term is the tangent vector  $M_{\gamma, 1} \partial_t$  to  $\hat{\omega}_\gamma$ . However, if  $\mu > 1$  then the tangent vector is zero.

To include the case  $\mu > 1$  consider the group  $\mathfrak{G}$  of germs at identity of analytic curves in the groupoid  $Diff(\tau)$ .  $\mathfrak{G}$  has natural filtration by order of tangency of the germ to the constant germ, i.e. by the order of the first non-zero term in its Taylor decomposition in  $\epsilon$ , which induces a filtration on its Lie algebra, and the associated graded algebra is a Lie algebra  $\hat{\mathfrak{G}}$  isomorphic to  $\mathcal{X} \otimes \mathbb{C}[[\epsilon]]$ , up to a shift of grading by 1. The homomorphism  $\mathfrak{R}$  pulls back the above filtration of  $\mathfrak{G}$  to a filtration of  $\pi_1$ , compatible with the group commutator (for generic perturbations this filtration most probably coincides with the lower central series  $L_i$ ). Starting from this filtration on  $\pi_1$ , one can build a Lie algebra in a standard way, and Proposition 3.4 shows that  $\mathfrak{R}$  lifts to a Lie algebra mapping between this Lie algebra and  $\hat{\mathfrak{G}}$ .

*Proof.* The monodromy of  $\gamma = \gamma_1 \gamma_2 \gamma_1^{-1} \gamma_2^{-1}$  is given by  $P_\gamma = P_{\gamma_2}^{-1} \circ P_{\gamma_1}^{-1} \circ P_{\gamma_2} \circ P_{\gamma_1}$ . Denote  $P_{\gamma_i}(t, \epsilon) = t + \epsilon^{\mu_i} M_{\gamma_i, \mu_i} + \dots + \epsilon^\mu M_{\gamma_i, \mu} + o(\epsilon^\mu)$  for  $i = 1, 2$ . Then

$$\begin{aligned} P_{\gamma_1} \circ P_{\gamma_2}(t, \epsilon) &= P_{\gamma_2}(t, \epsilon) + \sum_{j=\mu_1}^{\mu} \epsilon^j M_{\gamma_1, j}(P_{\gamma_2}(t, \epsilon)) + o(\epsilon^\mu) = \\ &= t + \sum_{j=\mu_2}^{\mu} \epsilon^j M_{\gamma_2, j}(t) + \sum_{j=\mu_1}^{\mu} \epsilon^j M_{\gamma_1, j}(t) + \epsilon^\mu M'_{\gamma_1, \mu_1} M_{\gamma_2, \mu_2} + o(\epsilon^\mu). \end{aligned}$$

Similarly,

$$P_{\gamma_2} \circ P_{\gamma_1}(t, \epsilon) = t + \sum_{j=\mu_1}^{\mu} \epsilon^j M_{\gamma_1, j}(t) + \sum_{j=\mu_2}^{\mu} \epsilon^j M_{\gamma_2, j}(t) + \epsilon^\mu M'_{\gamma_2, \mu_2} M_{\gamma_1, \mu_1} + o(\epsilon^\mu),$$

and therefore

$$P_{\gamma_2} \circ P_{\gamma_1}(t, \epsilon) = P_{\gamma_1} \circ P_{\gamma_2}(t, \epsilon) + \epsilon^\mu (M_{\gamma_1, \mu_1} M'_{\gamma_2, \mu_2} - M'_{\gamma_1, \mu_1} M_{\gamma_2, \mu_2}) + o(\epsilon^\mu).$$

As  $(P_{\gamma_2}^{-1} \circ P_{\gamma_1}^{-1})' = 1 + O(\epsilon)$ , application of  $(P_{\gamma_2}^{-1} \circ P_{\gamma_1}^{-1}(s, \epsilon), \epsilon)$  provides the required equality

$$P_\gamma = P_{\gamma_2}^{-1} \circ P_{\gamma_1}^{-1} \circ P_{\gamma_2} \circ P_{\gamma_1}(t, \epsilon) = t + \epsilon^\mu (M_{\gamma_1, \mu_1} M'_{\gamma_2, \mu_2} - M'_{\gamma_1, \mu_1} M_{\gamma_2, \mu_2}) + o(\epsilon^\mu). \quad \square$$

**3.4. Explicit computations.** By Poincaré-Pontryagin criterion,  $M_{\sigma, 1} = \int_\sigma \omega$ . From (3.1) we have

$$\begin{aligned} (2\pi i)^{-1} \int_{\delta_1 + \delta_2} \omega &= a_2(t), \\ (2\pi i)^{-1} \int_{\delta_2(t)} \omega &= a_2(t) - a_3(t), \\ (2\pi i)^{-1} \int_{\delta_2 + \delta_3} \omega &= a_1(t) - a_3(t). \end{aligned}$$

By Proposition 3.4 we have

$$\begin{aligned} (2\pi i)^{-2} M_{v_2, 2}(t) &= W(a_2(t), a_1(t) - a_3(t)), \\ (2\pi i)^{-3} M_{v_3, 3}(t) &= W(a_2(t), W(a_2(t) - a_3(t), a_1(t) - a_3(t))). \end{aligned} \quad (3.14)$$

In what follows, the  $2\pi i$  factors are not important, so we will omit them.

**Lemma 3.6.**  $M_{v_2, 2} \equiv 0$  and  $M_{v_3, 3} \neq 0$  if, and only if,

$$\begin{aligned} a_3(t) &= \alpha_1 \int_0^t \frac{\alpha_2(\tau)}{\alpha_1^2(\tau)} d\tau + c_0 \alpha_1 \\ a_1(t) &= a_3(t) + \alpha_1(t) \\ a_2(t) &= \lambda \alpha_1(t), \text{ with } \lambda \in \mathbb{C}^*, \end{aligned} \quad (3.15)$$

where  $\alpha_1(t)$  and  $\alpha_2(t)$  are linearly independent functions over  $\mathbb{C}$ , and  $\alpha_1$  is not constant.

*Proof.* From  $M_{v_3, 3} \neq 0$  we see that  $a_2(t), a_1(t) - a_3(t) \neq 0$ .

Then  $M_{v_2, 2} \equiv 0$  is equivalent to

$$a_2(t) = \lambda_1 (a_1(t) - a_3(t)), \text{ for some } \lambda_1 \in \mathbb{C}^*. \quad (3.16)$$

This implies, by linearity of Wronskians,

$$M_{v_3, 3} = W(a_2(t), W(a_3(t), a_1(t))) = \lambda_1 W(a_1(t) - a_3(t), W(a_3(t), a_1(t))). \quad (3.17)$$

Then,  $M_{v_3, 3} \neq 0$  if, and only if,

$$W(a_1, a_3) \neq \lambda_2 (a_1(t) - a_3(t)), \text{ for all } \lambda_2 \in \mathbb{C}. \quad (3.18)$$

In other words,

$$\begin{aligned} a_1(t) - a_3(t) &= \alpha_1(t) \\ a_3'(t)a_1(t) - a_1'(t)a_3(t) &= \alpha_2(t), \end{aligned} \quad (3.19)$$

where  $\alpha_1(t)$  and  $\alpha_2(t)$  are linearly independent functions over  $\mathbb{C}$ , and  $\alpha_1$  is not constant, in order to get condition (3.18).

The solution of the system (3.19) is

$$\begin{aligned} a_1(t) &= a_3(t) + \alpha_1(t) \\ a_3(t) &= \alpha_1 \int_0^t \frac{\alpha_2(\tau)}{\alpha_1^2(\tau)} d\tau + c_0 \alpha_1, \quad c_0 \in \mathbb{C}. \end{aligned}$$

Substituting  $a_1$  and  $a_3$  in expression (3.16) we get  $a_2(t)$ .  $\square$

In general,  $M_{v_2,2} \equiv 0$  does not necessarily imply  $M_{\gamma,2}(t) \equiv 0$ . However,

**Lemma 3.7.** *For a form  $\omega$  of form (3.2), the condition  $M_{v_2,2}(t) \equiv 0$  is equivalent to  $M_{\gamma,2}(t) \equiv 0$ .*

*Proof.* Evidently, if  $M_{\gamma,2}(t) \equiv 0$  then  $M_{v_2,2}(t) = \text{Var}(M_{\gamma,2}(t)) \equiv 0$ , so one implication is trivial.

Since  $M_{\gamma,1} \equiv 0$  by Françoise algorithm we have that  $M_{\gamma,2} = \int_{\gamma} \omega' \omega$ . Using integration by parts, we can rewrite  $\omega$  as

$$\omega = \sum_{i=1}^3 a_i(t) d\phi_i = - \sum_{i=1}^3 \phi_i a_i'(F) dF + d\left(\sum_{i=1}^3 a_i(F) \phi_i\right), \quad \phi_i = \log f_i.$$

Denote  $g = -\sum_{i=1}^3 \phi_i a_i'(F)$  and  $R = \sum_{i=1}^3 a_i(F) \phi_i$ . Then,  $\omega' = dg$  and so  $M_{\gamma,2} = \int_{\gamma(t)} g \omega$ . Developing this expression we get

$$M_{\gamma,2} = \sum_{1 \leq i < j \leq 3} W(a_i(t), a_j(t)) \int_{\gamma(t)} \phi_i d\phi_j. \quad (3.20)$$

Next Lemma is useful in following computations.

**Lemma 3.8.** *Assume that functions  $f_i(x)$ ,  $i = 1, \dots, m$ , are holomorphic in some simply connected domain  $U \subset \mathbb{C}_x$  containing the projection  $\gamma_x$  of  $\gamma$  to the  $x$ -axis  $\mathbb{C}_x$ . Then the iterated integral  $\int_{\gamma} (f_1 dx) \dots (f_m dx)$  vanishes.*

*Proof.* This integral is equal to  $\int_{\gamma_x} (f_1 dx) \dots (f_m dx)$ , so this follows from Cauchy theorem.  $\square$

By Lemma 3.8,  $\int_{\gamma(t)} \phi_1 d\phi_3 = \int_{\gamma(t)} \log(x+1) \frac{dx}{x-1} \equiv 0$ . On the other hand, since  $M_{v_2,2} = W(a_2(t), a_1(t) - a_3(t)) \equiv 0$ , we have  $W(a_1, a_2) + W(a_2, a_3) = 0$ , therefore

$$M_{\gamma,2} = W(a_1(t), a_2(t)) \left( \int_{\gamma(t)} \phi_1 d\phi_2 - \int_{\gamma(t)} \phi_2 d\phi_3 \right).$$

Substituting  $\phi_2 d\phi_3$  by  $-\phi_3 d\phi_2 + d(\phi_2 \phi_3)$ , and since  $\int_{\gamma(t)} d(\phi_2 \phi_3) = 0$ , we have

$$M_{\gamma,2} = W(a_1(t), a_2(t)) \left( \int_{\gamma(t)} \phi_1 d\phi_2 + \phi_3 d\phi_2 \right).$$

Recall that  $\phi_1 = \log(x+1)$ ,  $\phi_2 = \log(y-1)$  and  $\phi_3 = \log(x-1)$ , so

$$\begin{aligned} M_2(\gamma(t)) &= W(a_1(t), a_2(t)) \int_{\gamma(t)} (\log(x+1) + \log(x-1)) \frac{dy}{y-1} \\ &= W(a_1(t), a_2(t)) \int_{\gamma(t)} \log(x^2-1) \frac{dy}{y-1} \\ &= W(a_1(t), a_2(t)) \int_{\gamma(t)} \log\left(\frac{t}{y^2-1}\right) \frac{dy}{y-1}. \end{aligned}$$

Again, by Lemma 3.8 this integral vanishes.  $\square$

*Proof of Proposition 3.2.* Proposition 3.2 follows from Lemmas 3.6, 3.7.  $\square$

3.4.1.  $M_{\gamma,2} = M_{\gamma,3} = 0$  implies center.

**Proposition 3.9.** *For deformation (1.1) with  $\omega$  as in (3.2), identical vanishing of both  $M_{\gamma,2}, M_{\gamma,3}$  is equivalent to preservation of the center.*

One implication is trivial, so we assume  $M_{\gamma,2} = M_{\gamma,3} \equiv 0$  and prove that (1.1) preserves the center.

First, there is a trivial symmetric case.

**Lemma 3.10.** *If either  $a_2(t) \equiv 0$  or  $a_1(t) - a_3(t) \equiv 0$ , then (1.1) defines a center.*

*Proof.* Indeed, then the foliation (1.1) is symmetric with respect to the symmetry  $y \rightarrow -y$  or with respect to the symmetry  $x \rightarrow -x$ , correspondingly.  $\square$

Further, we assume that  $a_1(t) - a_3(t), a_2(t) \neq 0$ . The conditions  $M_{\gamma,2} = M_{\gamma,3} = 0$  imply  $M_{v_2,2} = M_{v_3,3} = 0$ . From (3.14)(3.17), this is equivalent to

$$\begin{aligned} a_1 - a_3 &= \lambda_1 a_2, & \lambda_1 &\in \mathbb{C}^* \\ W(a_1, a_3) &= \lambda_2 a_2, & \lambda_2 &\in \mathbb{C}. \end{aligned} \quad (3.21)$$

This implies

$$W(a_1, a_2) = -a_2^2 \left(\frac{a_1}{a_2}\right)' = -\lambda a_2, \quad \lambda = \frac{\lambda_2}{\lambda_1}, \quad (3.22)$$

i.e. necessarily

$$\begin{aligned} a_1 &= a_2 \left( \lambda \int \frac{dt}{a_2} + c_1 \right) \\ a_3 &= a_2 \left( \lambda \int \frac{dt}{a_2} + c_1 - \lambda_1 \right). \end{aligned} \quad (3.23)$$

Denote  $A(t) = \int \frac{dt}{a_2}$ , so

$$\omega = a_2(t) [A(t)\alpha + d\phi], \quad \alpha = \lambda(d\phi_1 + d\phi_3), \quad \phi = c_1\phi_1 + \phi_2 + (c_1 - \lambda_1)\phi_3. \quad (3.24)$$

As  $a_2(t) = A'(t)^{-1}$ , the foliation (1.1) is then equal to

$$dF + \epsilon \frac{1}{A'(F)} (A(F)\alpha + d\phi) = 0, \quad (3.25)$$

which is orbitally equivalent to

$$dA(F) + \epsilon\eta = 0, \quad \eta = A(F)\alpha + d\phi. \quad (3.26)$$

If  $\lambda = 0$ , then  $\alpha = 0$ , and this is a Hamiltonian system with Hamiltonian  $A(F) + \epsilon\phi$  (recall that  $\phi$  is a holomorphic function in a neighborhood of  $\gamma$ ). Thus  $\lambda = 0$  implies preservation of center, and it remains to prove that vanishing of  $M_{\gamma,3}$  implies  $\lambda = 0$ .

We consider (3.26) as a perturbation of a Hamiltonian system with Hamiltonian  $A(F)$ .

**Lemma 3.11.** *The first non-zero Melnikov functions  $M_{\gamma,k}$  of (3.25) and the first non-zero Melnikov functions  $\tilde{M}_{\gamma,k}$  of (3.26) are related by*

$$\tilde{M}_k = g'(F)M_k. \quad (3.27)$$

*Proof.* Indeed, the passage from (3.25) to (3.26) amounts to reparameterization of the transversal  $\tau$  by values of  $A(F)$ . For a point  $p \in \tau$  with  $F(p) = t$ ,

$$\tilde{M}_{\gamma,k} = \frac{\partial A(F(P(\epsilon, p)))}{\partial \epsilon^k} = A'(t) \frac{\partial F(P(\epsilon, p))}{\partial \epsilon^k} = A'(t)M_{\gamma,k}. \quad \square$$

*Remark 3.12.* In other words, the first non-zero Melnikov function has tensor type of a vector field, which is expected from Proposition 3.4 and the following Remark.

*Proof of Proposition 3.9.* We will compute  $\tilde{M}_{\gamma,3}$ . By Françoise's algorithm [2, 3], we have that

$$\tilde{M}_{\gamma,3} = \int_{\gamma} (\eta' \eta)' \eta, \quad (3.28)$$

where the Gelfand-Leray derivative is taken with respect to the Hamiltonian  $A(F)$ . Developing the derivative,

$$\tilde{M}_{\gamma,3} = \int_{\gamma} \eta'' \eta \eta + \int_{\gamma} \eta' \eta' \eta + \int_{\gamma} \frac{\eta' \wedge \eta}{dA} \eta, \quad (3.29)$$

where  $\eta' = \frac{d(A\alpha + d\phi)}{dA} = \alpha$ , and  $\eta'' = 0$ , as the forms  $\alpha, d\phi$  are closed. Then,

$$\tilde{M}_{\gamma,3} = \int_{\gamma} \alpha \alpha (A\alpha + d\phi) + \int_{\gamma} \frac{\alpha \wedge (A\alpha + d\phi)}{dA} \eta. \quad (3.30)$$

By Lemma 3.8 the triple integrals  $\int_{\gamma} \alpha \alpha d\phi_1, \int_{\gamma} \alpha \alpha d\phi_3$  vanish. Also, as

$$\alpha = \lambda d \log F - \lambda d \log(y^2 - 1), \quad (3.31)$$

Lemma 3.8 implies that the integral  $\int_{\gamma} \alpha \alpha d\phi_2$  vanishes.

Hence,

$$\tilde{M}_{\gamma,3} = \int_{\gamma} \frac{\alpha \wedge (A\alpha + d\phi)}{dA} \eta = \int_{\gamma} \frac{\alpha \wedge d\phi}{dA} \eta. \quad (3.32)$$

We have  $\alpha \wedge d\phi = \alpha \wedge d\phi_2$ . By (3.31), we have  $\alpha \wedge d\phi_2 = \frac{\lambda dF}{F} \wedge d\phi_2$ , so

$$\frac{\alpha \wedge d\phi}{dA} = \frac{\lambda \frac{dF}{F} \wedge d\phi_2}{A'(F)dF} = \frac{\lambda}{FA'(F)} d\phi_2, \quad (3.33)$$

and, as  $\eta = A(F)\alpha + d\phi$ ,

$$\tilde{M}_{\gamma,3} = \frac{\lambda}{tA'(t)} \int_{\gamma} d\phi_2 \eta = \frac{\lambda}{A'(t)} \int_{\gamma} d\phi_2 \alpha + \frac{\lambda}{tA'(t)} \int_{\gamma} d\phi_2 d\phi. \quad (3.34)$$

Again,  $\int_{\gamma} d\phi_2 \alpha \equiv 0$  by (3.31) and Lemma 3.8. Moreover,

$$\int_{\gamma} d\phi_2 d\phi = \int_{\gamma} d\phi_2 d\phi_2 + \frac{c_1}{\lambda} \int_{\gamma} d\phi_2 \alpha - \lambda \int_{\gamma} d\phi_2 d\phi_3 = -\lambda \int_{\gamma} d\phi_2 d\phi_3,$$

as  $\int_{\gamma} d\phi_2 d\phi_2 \equiv 0$  by Lemma 3.8. Therefore

$$\tilde{M}_{\gamma,3} = \frac{-\lambda^2}{tA'(t)} \int_{\gamma} d\phi_2 d\phi_3. \quad (3.35)$$

We claim that

$$\int_{\gamma} d\phi_2 d\phi_3 \neq 0. \quad (3.36)$$



Indeed, from  $\int_{\gamma} d\phi_i \equiv 0$  and  $\int_{[\sigma_1, \sigma_2]} \omega_1 \omega_2 = \det \left\{ \int_{\sigma_i} \omega_j \right\}_{i,j=1}^2$ , we see that  $\int d\phi_2 d\phi_3$  vanishes on  $K$ , and, therefore, defines a linear functional on  $(O/K)^*$ . Therefore,

$$\text{Var}^2 \int_{\gamma} d\phi_2 d\phi_3 = \int_{v_2} d\phi_2 d\phi_3 = \det \begin{pmatrix} \int_{\delta_1+\delta_2} d\phi_2 & \int_{\delta_1+\delta_2} d\phi_3 \\ \int_{\delta_2+\delta_3} d\phi_2 & \int_{\delta_2+\delta_3} d\phi_3 \end{pmatrix} = 4\pi^2$$

by (3.1), which proves (3.36).

Thus  $\tilde{M}_{\gamma,3}$ , as well as  $M_{\gamma,3}$ , vanish identically only if  $\lambda = 0$ , which finishes the proof of Proposition 3.9.  $\square$

*Proof of Theorem 1.6.* Take  $\omega$  as in (1.7), with coefficients  $a_i(F)$  as in Proposition 3.2. The first Melnikov function  $M_{\gamma,1}$  vanishes identically for all  $\omega$  of this type. Also,  $M_{\gamma,2}$  vanishes by Proposition 3.2. Therefore  $M_{\gamma,3}$  is the first non-zero Melnikov function of (1.1) or it is identically zero. In both cases, it is linear on the orbit, see [4, 7], and therefore  $\text{Var}^3(M_{\gamma,3}(t)) = M_{v_3,3}(t) \neq 0$ . Therefore  $M_{\gamma,3}(t) \neq 0$ , and, moreover, has length three. Taking  $\alpha_1 = t$ ,  $\alpha_2 = t^2$  and  $c_0 = \lambda = 1$ , one gets the example of Theorem 1.6.

The last statement of Theorem 1.6 follows from Proposition 3.9.  $\square$

**3.5. Length bigger than 4.** Now, for deformation (1.1) consider the functions  $M_{v_i,i}$ , where  $v_i$  were defined in Proposition 2.7. Note that  $M_{v_i,j}$ , given by iterated integrals of length at most  $j$ , necessarily vanish on  $v_i \in L_i$  for  $j < i$ , so  $M_{v_i,i}$  are (generically) the first non-zero Melnikov functions of  $v_i$  with respect to the deformation (1.1).

**Lemma 3.13.** *The condition  $M_{v_2,2} = M_{v_3,3} = 0$  implies  $M_{v_i,i} = 0$  for all  $i \geq 4$ .*

*Remark 3.14.* Vanishing of  $M_{v_i,i}$  is necessary for vanishing of  $M_{\gamma,i}$  (i.e. follows from center conditions), but not sufficient, see for example Proposition 3.9.

*Proof.* Denote  $\beta_1 = \int_{\delta_1+\delta_2} \omega$ ,  $\beta_2 = \int_{\delta_2} \omega$  and  $\beta_3 = \int_{\delta_2+\delta_3} \omega$ . By Proposition 3.4,

$$\begin{aligned} M_{v_2,2} &= W(\beta_1, \beta_3) \\ M_{v_3,3} &= W(\beta_1, W(\beta_2, \beta_3)) \\ M_{v_4,4} &= W(\beta_1, W(\beta_2, W(\beta_2, \beta_3))) \\ &\vdots \\ M_{v_i,i} &= W(\beta_1, W(\beta_2, \dots, W(\beta_2, \beta_3)) \dots). \end{aligned}$$

Suppose  $M_{v_2,2} \equiv 0$ . If  $\beta_3 \equiv 0$  then evidently all these Wronskians vanish. Otherwise,  $\beta_1 = \lambda_1 \beta_3$ , for some  $\lambda \in \mathbb{C}$ .

Suppose also that  $M_3(v_3) \equiv 0$ . Again, the case  $\beta_1 \equiv 0$  is trivial. Otherwise,  $W(\beta_2, \beta_3) = \lambda_2 \beta_1$ , for some  $\lambda_2 \in \mathbb{C}$ , and therefore

$$W(\beta_2, \beta_3) = \lambda_1 \lambda_2 \beta_3,$$

which implies

$$M_{v_{i+1},i+1} = \lambda_1 \lambda_2 M_{v_i,i} = \dots = (\lambda_1 \lambda_2)^{i-2} M_{v_3,3} \equiv 0. \quad \square$$

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