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Packing colorings of subcubic outerplanar graphs

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Abstract

Given a graph G and a nondecreasing sequence $S = (s_1, \dots, s_k)$ of positive integers, the mapping $c : V(G) \rightarrow \{1, \dots, k\}$ is called an S -packing coloring of G if for any two distinct vertices x and y in $c^{-1}(i)$, the distance between x and y is greater than s_i . The smallest integer k such that there exists a $(1, 2, \dots, k)$ -packing coloring of a graph G is called the packing chromatic number of G , denoted $\chi_\rho(G)$. The question of boundedness of the packing chromatic number in the class of subcubic (planar) graphs was investigated in several earlier papers; recently it was established that the invariant is unbounded in the class of all subcubic graphs.

In this paper, we prove that the packing chromatic number of any 2-connected bipartite subcubic outerplanar graph is bounded by 7. Furthermore, we prove that every subcubic triangle-free outerplanar graph has a $(1, 2, 2, 2)$ -packing coloring, and that there exists a subcubic outerplanar graph with a triangle that does not admit a $(1, 2, 2, 2)$ -packing coloring. In addition, there exists a subcubic triangle-free outerplanar graph that does not admit a $(1, 2, 2, 3)$ -packing coloring. A similar dichotomy is shown for bipartite outerplanar graphs: every such graph admits an S -packing coloring for $S = (1, 3, \dots, 3)$, where 3 appears Δ times (Δ being the maximum degree of vertices), and this property does not hold if one of the integers 3 is replaced by 4 in the sequence S .

Keywords: outerplanar graph; packing chromatic number; cubic graph; coloring; packing.

AMS Subj. Class. (2010): 05C15, 05C12, 05C70

1 Introduction

The S -packing chromatic number was introduced a decade ago in [17] with motivation coming from the frequency assignment problem. Roughly the idea of the concept is to generalize the classical coloring by involving the distance between vertices and allowing larger color values only for vertices that are more distant. Nevertheless, the problem has attracted the attention of many discrete mathematicians as it brings appealing combinatorial and computational challenges.

Given a graph G and a positive integer d , a set $A \subseteq V(G)$ is a d -packing in G if for any two distinct vertices $x, y \in A$ the distance between x and y in G is greater than d . For a nondecreasing sequence $S = (s_1, \dots, s_k)$ of positive integers, the mapping $c : V(G) \rightarrow \{1, \dots, k\}$ is an S -packing coloring of G if for every $i \in [k]$ the set $c^{-1}(i)$ is an s_i -packing. If there exists an S -packing coloring of G , we say that G is S -packing colorable. If the sequence is $S = [k]$ for some positive integer k , we omit S in the definition, and say that G is packing colorable (as usual, we let $[k] = \{1, \dots, k\}$). The smallest integer k such that G is packing colorable is the packing chromatic number of G , denoted $\chi_\rho(G)$. When we say that the packing coloring condition holds for a set $A \subseteq V(G)$ we mean that each set $A \cap c^{-1}(i)$, for all $i \geq 1$, is an i -packing in G . If $A \subset V(G)$, then by $G[A]$ we denote the subgraph of G induced by A .

A number of papers considered packing coloring of different infinite grids and lattices [7, 13, 14, 19, 24], where the most interesting development is about the infinite square grid; we mention only the recent paper on the topic [5] where it was shown that $13 \leq \chi_\rho(\mathbb{Z} \times \mathbb{Z}) \leq 15$, which is the latest refinement of the known bounds (initial bounds were presented already in the seminal paper [17]). Fiala and Golovach have shown that the decision version of the packing chromatic number is NP-complete even in the class of trees [12]. Packing coloring of some other classes of graphs, such as distance graphs [11, 22, 25], hypercubes [26], subdivision graphs of subcubic graphs [4, 10, 15], and some other classes of graphs [2, 18, 20] was also studied.

One of the main questions in this area concerns graphs with bounded maximum degree Δ , in particular, *subcubic graphs* (i.e., graphs with $\Delta = 3$). For graphs with maximum degree Δ , where $\Delta \geq 4$, the infinite Δ -regular tree serves as an example showing that in this class of graphs the packing chromatic number is unbounded (in fact, Sloper proved this in the context of so-called eccentric colorings, but his result implies the same for the packing coloring [23]). On the other hand, the question whether in subcubic graphs the packing chromatic number is bounded was much more intriguing. It was posed in the seminal paper [17], and then investigated in several papers [8, 9, 15] using different approaches. Recently, Balogh, Kostochka and Liu [3] have provided a negative answer to the question. Moreover, they proved that for every fixed k and $g \geq 2k + 2$, almost every n -vertex cubic graph of girth at least g has the packing chromatic number greater than k . An explicit infinite family of subcubic graphs with unbounded packing chromatic number was then presented in [6].

As the question was answered in the negative for all graphs with bounded

maximum degree 3, it becomes interesting for some subclasses of subcubic graphs. In particular, in [13] it was asked, whether there is an upper bound for the packing chromatic number of all planar cubic graphs, and this question was repeated in [6]. Very recently, the packing chromatic number of subcubic outerplanar graphs was considered [16]. The upper bounds obtained in the paper involve the number of (internal) faces of the plane embedding of an outerplanar graph; for instance, it is proven that if G is a 2-connected subcubic outerplanar graph with r internal faces, then $\chi_\rho(G) \leq 17 \cdot 6^{3r} - 2$. The question of boundedness of the packing chromatic number in subcubic outerplanar graphs thus seems widely open. In this paper, we prove that, quite surprisingly, only 7 colors suffice if we restrict ourselves to the bipartite 2-connected case.

In the following section we fix the notation. In Section 3 we prove the following theorem, our main result.

Theorem 1. *Let G be a 2-connected bipartite subcubic outerplanar graph. Then $\chi_\rho(G) \leq 7$.*

We continue in Sections 4 and 5 with some results that are related to the coloring of the square of a graph. It was proven by Lih and Wang [21] that $\chi(G^2) \leq \Delta(G) + 2$ for an outerplanar graph G (see also [1] for an extension), which confirms Wegner's old conjecture for planar graphs in the case of outerplanar graphs. In the language of S -packing colorings the result in [21] for outerplanar graphs G with $\Delta(G) \leq 3$ implies that G is $(2, 2, 2, 2, 2)$ -packing colorable. More generally, for an arbitrary $\Delta(G) \geq 3$, the result of Lih and Wang [21] gives the $(2, \dots, 2)$ -packing colorability of outerplanar graphs G , where there are $\Delta(G) + 2$ integers 2.

In Section 4, we present the following result about bipartite outerplanar graphs (i.e., no restriction to 2-connectedness and arbitrary maximum degree).

Theorem 2. *Let G be a bipartite outerplanar graph. Let $S = (1, 3, \dots, 3)$ be the sequence containing once the integer 1 and k times the integer 3, $k \geq 3$. If $\Delta(G) \leq k$, then G is S -packing colorable.*

The result is complemented by an example showing that for $S = (1, 3, \dots, 3, 4)$, where 3 appears $\Delta(G) - 1$ times, there exists a bipartite outerplanar graph that does not admit an S -packing coloring.

In Section 5, subcubic outerplanar graphs are considered (extending the consideration of Theorem 1 to the non-bipartite case), and we prove:

Theorem 3. *If G is a subcubic outerplanar graph with no triangles, then G is $(1, 2, 2, 2)$ -packing colorable.*

The result is complemented by an example showing that there exists a subcubic outerplanar graph (with triangles), which is not $(1, 2, 2, 2)$ -packing colorable. In addition, there exists a subcubic triangle-free outerplanar graph that does not admit a $(1, 2, 2, 3)$ -packing coloring.

In the final section, we present a variation of Theorem 3 concerning the $(1, 1, 2)$ -packing colorability of subcubic triangle-free outerplanar graphs, and pose some open problems.

2 Notation

A path between vertices a and b in a graph G will be called an a, b -path. The length of a shortest a, b -path is the *distance* $d_G(a, b)$ between a and b in G (we also write $d(a, b)$ if the graph is understood from the context). An i -packing in G is a set of vertices A such that for any two distinct vertices $x, y \in A$ we have $d_G(x, y) > i$. Clearly, a 1-packing coincides with an independent set.

An outerplanar graph is a graph that has a planar drawing in which all vertices belong to the outer face of the drawing. Each time an outerplanar graph is considered, a drawing in which all vertices belong to the outer face of the drawing will be fixed. Let G be an outerplanar graph. The *outer cycle* of G corresponds to the cycle induced by the edges of the outer face.

For an outerplanar graph G , we denote by \mathcal{T}_G the *weak dual* of G , i.e., the graph whose vertex set is the set of all inner faces of G , and $E(\mathcal{T}_G) = \{\alpha\beta \mid \alpha \text{ and } \beta \text{ share a common edge}\}$. For $\alpha \in V(\mathcal{T}_G)$, we denote by $C(\alpha)$, the (chordless) cycle in G that corresponds to the face α . As any vertex $\alpha \in \mathcal{T}_G$ corresponds to an inner face of G , we let α also denote this face and write $V(\alpha)$ for the set of its vertices.

Let G be a 2-connected outerplanar graph (we consider such graphs in Section 3). Note that in this case \mathcal{T}_G is a tree. We consider \mathcal{T}_G as a rooted tree with an arbitrary chosen vertex ω_0 as the root. The notions of *parent*, *child* and *descendant* should be clear in this context. We can also define the *depth* of a vertex $\beta \in V(\mathcal{T}_G)$, denoted by $p(\beta)$, as $d_{\mathcal{T}_G}(\beta, \omega_0)$. The depth of \mathcal{T}_G , denoted by $p(\mathcal{T}_G)$, is the maximum value of $p(\beta)$, for $\beta \in V(\mathcal{T}_G)$.

3 Packing coloring of 2-connected bipartite subcubic outerplanar graphs

In this section we prove that a 2-connected bipartite subcubic outerplanar graph G has packing chromatic number bounded by 7, i.e., $\chi_\rho(G) \leq 7$. At the end of the section we add a result (Proposition 5) which shows that this is best possible.

The proof of the main theorem has two steps. In the first step we construct a subset B of $V(G)$, and present a coloring f of the vertices of $A = V(G) \setminus B$ by using only the colors from $\{1, 2, 3\}$ such that the packing coloring condition holds for A . The set B will be called the set of *big vertices*, and so any vertex in B is called a *big vertex*. The big vertices will be colored in the second step by using only the colors from $\{4, 5, 6, 7\}$. That is, we will extend f from A to all vertices of $V(G)$. We will prove that $f : V(G) \rightarrow \{1, \dots, 7\}$ has four special properties, which will be helpful in proving that f is a packing coloring of G (that is, for any two distinct vertices $u, v \in V(G)$, $f(u) = f(v)$ implies $d_G(u, v) > f(u)$). As in many cases, additional (technical) conditions, which are not needed in the result, are helpful in the proof.

Proof of Theorem 1.

Let G be a 2-connected bipartite subcubic outerplanar graph. Let ω_0 be a vertex of \mathcal{T}_G of minimum eccentricity; we consider \mathcal{T}_G to be a rooted tree with ω_0 as its root.

In the proof we will construct a set of vertices B and a packing coloring f of G . The coloring f will satisfy the following additional properties.

- (i) Any vertex with color different from 1 (including big vertices whose color will be determined in Step 2) has all its neighbors colored by color 1.
- (ii) Any face α of G contains exactly one big vertex if $|\alpha| \geq 6$ and at most one big vertex if $|\alpha| = 4$.
- (iii) Any big vertex is at distance at least 4 from any other big vertex.
- (iv) Any vertex with color from $\{6, 7\}$ is at distance at least 6 from any vertex with color from $\{6, 7\}$.

Step 1. In this first step of the proof, we construct the set B and consequently the set $A = V(G) \setminus B$, and color the vertices of A by using only the colors from $\{1, 2, 3\}$. During this step, we will ensure that properties (i), (ii) and (iii) are satisfied (while property (iv) will be verified in the second step, when we assign colors to the vertices from B).

The proof uses the structure of the tree \mathcal{T}_G . We consider the faces in a Breadth-first search (BFS) order by starting with the face ω_0 . In each facial cycle we will repeatedly use the pattern 1, 2, 1, 3; by this we mean that vertices along the cycle will follow in the order 1, 2, 1, 3, ... or 1, 3, 1, 2, ..., which also applies when the length of the cycle is not divisible by 4 (in which case, we omit the appropriate number of colors at the end of the sequence). If the length of the corresponding cycle is greater than 4, one of the vertices is taken as a big vertex and is not yet colored (it will not belong to the set A), and for the rest of the cycle we use the pattern 1, 2, 1, 3 (by repeating it an appropriate number of times). In a 4-cycle we may either use only the colors from $\{1, 2, 3\}$ or declare one vertex as big, which depends on the type of the used coloring (described soon).

Clearly, all the vertices of ω_0 can be colored by using the above pattern. In particular, if $p(\mathcal{T}_G) = 0$, then G is the cycle $C(\omega_0)$, so that we can set $A = V(\omega_0)$, and the described coloring satisfies properties (i), (ii), (iii) and (iv).

Suppose now $p(\mathcal{T}_G) > 0$. By following a BFS order of \mathcal{T}_G , consider a face $\alpha \in \mathcal{T}_G$, $\alpha \neq \omega_0$, where $\omega \in \mathcal{T}_G$ is the parent of α . By the construction, a big vertex u of ω is already determined (including the possibility that ω has no big vertices, which may happen when the size of $C(\omega)$ is 4), and other vertices of ω are colored by colors $\{1, 2, 3\}$ repeatedly using the pattern 1, 2, 1, 3.

We consider the following three cases, depending on the position of the big vertex in α (and the coloring of the vertices of A), respectively called 0-, 1- and 2-position, and extend the function f to all vertices of $\alpha \cap A$ in each of these cases, referred to as a 0-, 1- and 2-coloring, respectively.

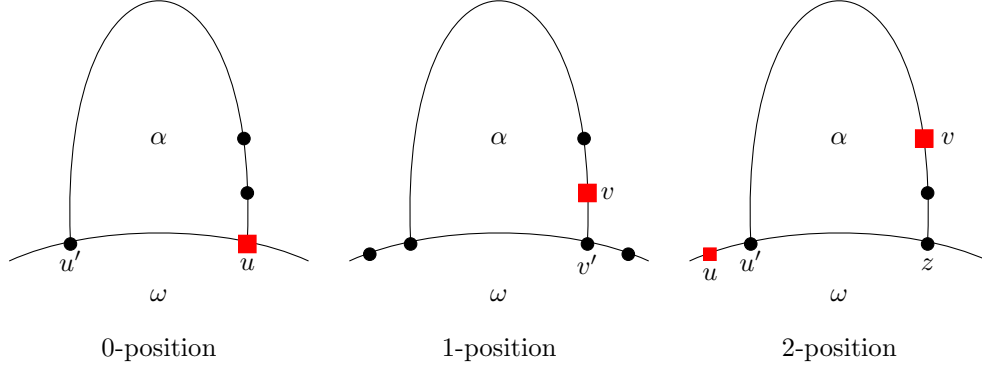


Figure 1: 0- 1- and 2-position in face α (circle: vertex of A ; square: vertex of B , i.e. big vertex).

- 0-position:** suppose that the big vertex u of ω coincides with one of the two vertices belonging to $\alpha \cap \omega$. In this case, the big vertex of α is also determined, notably, $u \in B \cap \alpha$. Let u' be the other vertex belonging to $\alpha \cap \omega$. (See Figure 1.) Thanks to property (i), u' is already colored by 1. The corresponding **0-coloring** of the vertices of $C(\alpha)$ is obtained by starting with the uncolored neighbor of u , and repeatedly using the pattern 1, 2, 1, 3 along the cycle (or, as mentioned before when describing the setting with the pattern 1, 2, 1, 3, the actual pattern may also be 1, 3, 1, 2 depending on the color of the neighbor of u' belonging to $\omega \setminus \alpha$), ensuring that the neighbors of u' get different colors; note that their colors will be k , 2 and 3, respectively, k being the color that u will get in step 2.
- 1-position:** suppose that the big vertex of ω is not one of the vertices belonging to $\alpha \cap \omega$ and, in addition, not adjacent to any of the two vertices belonging to $\alpha \cap \omega$. (In other words, the four vertices of $C(\omega)$ that are closest to $C(\alpha)$ are colored by colors 1, 2, 1 and 3.) Let v' be the vertex belonging to $\alpha \cap \omega$, for which $f(v') = 1$. And thus other big vertices in ω are at distance at least three from v' by (i). In this case, we define the big vertex v of α (i.e. the vertex v of α belonging to B) as the neighbor of v' , which does not lie in ω . (See Figure 1.) The corresponding **1-coloring** of the vertices of $C(\alpha)$ is obtained by starting with the neighbor of v' belonging to $\alpha \cap \omega$, and repeatedly using the pattern 1, 2, 1, 3 along the cycle.
- 2-position:** suppose that the big vertex u of ω is not one of the vertices belonging to $\alpha \cap \omega$, but is adjacent to one of the two vertices belonging to $\alpha \cap \omega$. Let us denote by u' the neighbor of u belonging to $\alpha \cap \omega$, and

let the other vertex belonging to $\alpha \cap \omega$ be called z . By (i), $f(u') = 1$, and $f(z) \in \{2, 3\}$. Now, if $C(\alpha)$ has 4 vertices, then we use only colors 1, 2 or 1, 3 ensuring that the neighbors of u' in $C(\alpha)$ get distinct colors. Otherwise, we define the big vertex v of α (i.e. the vertex v of α belonging to B) to be the vertex at distance 2 from the vertex z , different from a neighbor of u' . (See Figure 1.) The corresponding **2-coloring** of the vertices of $C(\alpha)$ is obtained by starting with u' , and repeatedly using the pattern 1, 2, 1, 3 along the cycle, ensuring that the neighbors of u' get different colors in $\{2, 3\}$.

Clearly, for any of the three colorings (0-coloring, 1-coloring, 2-coloring) the packing coloring condition holds for the vertices of A belonging to a face α following ω in the BFS order of \mathcal{T}_G , and properties (i), (ii) and (iii) extend from vertices colored so far to the vertices of α . We derive the following observation.

Lemma 4. *Let B be the set constructed in Step 1 with $A = V(G) \setminus B$, and let f be the coloring of the vertices of A by colors $\{1, 2, 3\}$ as described above. Then f and B satisfy properties (i), (ii) and (iii), and for any two distinct vertices x and y in A such that $f(x) = f(y)$ we have $d_G(x, y) > f(x)$.*

Step 2. In this step we need to determine the f -values of big vertices, and prove that property (iv) holds for these vertices and that the packing coloring condition holds for the set B . We determine the colors of big vertices following a BFS order on \mathcal{T}_G ; we start by determining the possible color of the big vertex of ω_0 . If such a vertex exists (i.e., $|V(\omega_0)| > 4$), then we color it by 4.

A big vertex x that belongs to a face β will be called a big vertex *arising* from α if the following two conditions are true:

- i) β is a descendant of α with respect to \mathcal{T}_G ;
- ii) x is at distance at most two from $C(\alpha)$.

A step of the coloring construction consists in dealing with a face α that comes next in the chosen BFS order (beginning with the face ω_0), and coloring all the big vertices arising from α . Let ω be the parent of the face α (if it exists). In a step of the coloring construction, it is supposed that, if $\alpha \neq \omega_0$, then both the big vertex of ω and the big vertices arising from ω are already colored. Consequently, the big vertex of α is already colored, because it arises from ω . On the other hand, when $\alpha = \omega_0$, only the big vertex of ω_0 is already colored.

We assume that f satisfies the packing coloring condition for big vertices colored in previous steps, and that property (iv) is satisfied for the big vertices colored in previous steps. Consequently, we have to prove that the packing coloring condition holds for big vertices and that property (iv) remains true when we extend the function f in a new step. We distinguish three cases with respect to the type of the position of the big vertex (0-position, 1-position, 2-position) and the corresponding coloring, used to color the big vertex of α . We will call the colors in $\{6, 7\}$ *very big*.

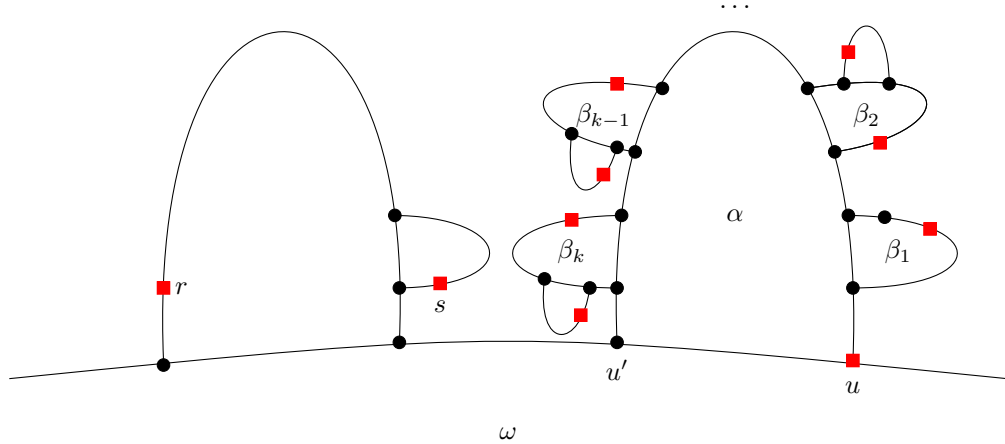


Figure 2: Big vertices arising from face α in 0-position (circle: vertex of A ; square: vertex of B , i.e. big vertex).

Case 1. α is in 0-position, see Figure 2.

Consider the face ω and its descendants with respect to \mathcal{T}_G , and note that the color of their big vertices could already have been determined (with the exception of big vertices arising from α). By property (i), all big vertices are at even distance from vertex u , and they are at distance at least 4 from u by property (iii). We distinguish two kinds of big vertices that were already colored, namely those that are at a shorter distance from u than from u' , and those that are at a shorter distance from u' than from u . Those that are closer to u than to u' are at distance at least 8 from big vertices that arise from α . (As shown in Figure 2, big vertices arising from α belong to faces β_1, \dots, β_k , which are children of α with respect to \mathcal{T}_G , or to their children.) On the other hand, there can be big vertices, which are at distance 3 from u' (and 4 from u), and have already been colored. More precisely, by using property (iii), and the fact that the neighbor of u' that is not in α has at most two other neighbors, we find that there can be at most two such big vertices, which are at distance 3 from u' . If they indeed exist, we denote them by r and s , and note that r, s and u are pairwise at distance 4 from each other.

Next we analyze possible positions of vertices that arise from α . For every vertex a in $C(\alpha)$, which is at an even distance from u , there can be a big vertex a' that arises from α such that there is an a, a' -path of length 2 outside α (there is at most one such vertex by property (iii)). On the other hand, for every vertex b in $C(\alpha)$, which is at an odd distance from u , there can be a big vertex b' that arises from α , which is adjacent to b (with the exception of the neighbors of u in $C(\alpha)$, which cannot be adjacent to another big vertex due to property

(iii)). The situation when all these big vertices exist is described in Figure 3.

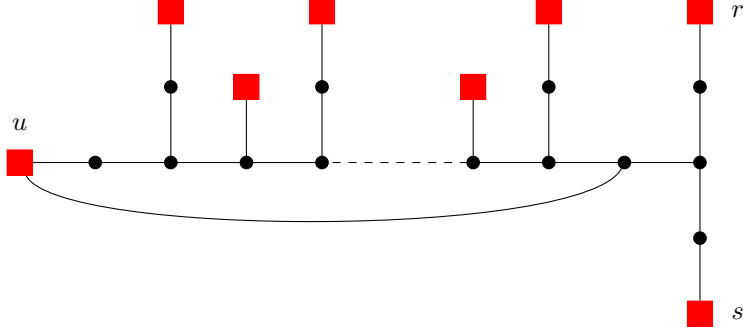


Figure 3: Case 1 (α in 0-position).

In the most complex case when both vertices r and s exist and are already colored, vertices u , r and s are pairwise at distance 4 from each other. By property (iv), exactly one of these three vertices is colored by a very big color. We consider two subcases, depending on the color of vertices u , r and s . Note that we can easily modify our coloring function so that all vertices colored by color 4 are colored by color 5 and vice-versa. Also, the same holds for the vertices colored by color 6 and 7 (we can exchange the two color classes). A consequence is that the way to deal with the case $f(u) = 5$, $\{f(r), f(s)\} = \{4, 7\}$ is the same as when $f(u) = 4$, $\{f(r), f(s)\} = \{5, 7\}$ or when $f(u) = 5$, $\{f(r), f(s)\} = \{4, 6\}$ and that the way to deal with the case $f(u) = 7$, $\{f(r), f(s)\} = \{4, 5\}$ is the same as when $f(u) = 6$, $\{f(r), f(s)\} = \{4, 5\}$. Thus, it suffices to deal only with two subcases: $f(u) = 4$, $\{f(r), f(s)\} = \{5, 7\}$ and $f(u) = 6$, $\{f(r), f(s)\} = \{4, 5\}$.

Subcase 1.a. $f(u) = 4$ and $\{f(r), f(s)\} = \{5, 7\}$. We present three patterns that define the coloring of the big vertices with respect to different lengths n of the cycle $C(\alpha)$. The patterns give f -values of the big vertices following their presentation in Figure 3 (from left to right); in the case when some of the big vertices that arise from α do not exist, we simply skip the corresponding values in the pattern.

If $n = 4$, there is at most one big vertex that arises from α , and we color it by 5. Now, assume that $n \geq 6$. Note that the numbers between vertical bars are to be repeated k times (case $k = 0$ included).

$$\text{Pattern for length } n = 4k + 8: \quad \begin{array}{c} 7 \\ 5 \end{array} \left| \begin{array}{c} 6 \\ 4 \end{array} \quad \begin{array}{c} 7 \\ 5 \end{array} \left| \begin{array}{c} 4 \\ 6 \end{array} \quad \begin{array}{c} 5 \end{array} \right.$$

$$\text{Pattern for length } n = 8k + 6: \quad \begin{array}{c} 5 \\ 6 \end{array} \left| \begin{array}{c} 5 \\ 4 \end{array} \quad \begin{array}{c} 5 \\ 7 \end{array} \quad \begin{array}{c} 5 \\ 4 \end{array} \quad \begin{array}{c} 5 \\ 6 \end{array} \left| \begin{array}{c} 5 \end{array} \right.$$

Pattern for length $n = 8k + 10$:

$$\begin{array}{cccc|cccc|c} 5 & & 5 & & 5 & & 5 & & 5 & & 5 & & 5 & & 5 \\ & 7 & & 4 & & 6 & & 4 & & 7 & & 4 & & 6 & & 5 \end{array}$$

Note that because $f(u) = 4$, the first and the last two values in the patterns cannot be 4, and because $\{f(r), f(s)\} = \{5, 7\}$, the last two values in the patterns cannot be 7. Also note that two vertices that correspond to two successive values in the upper row of the pattern are at distance 6, two vertices that correspond to two successive values in the lower row of the pattern are at distance 4, and two vertices that correspond to two consecutive values in the patterns (one in the upper and the other in the lower row) are at distance 4.

In each of the patterns, one can check that property (iv) holds, and that for any two identical numbers, the distance between the corresponding vertices in G is bigger than this number (i.e., the packing coloring condition is satisfied). Note that for $k = 0$ one needs to omit the numbers that are between the vertical bars in each pattern, which covers the lengths of cycles n , where $n \in \{6, 8, 10\}$.

Subcase 1.b. $f(u) = 6$ and $\{f(r), f(s)\} = \{4, 5\}$. If $n = 4$ we can use color 4 for the only big vertex that possibly arises from α . Now, assume that $n \geq 6$. We present patterns that define the coloring of the big vertices in the case $f(u) = 6$, and $\{f(r), f(s)\} = \{4, 5\}$. Note that it implies that the first and the last four values in the patterns cannot be 6. Again the numbers between vertical bars are to be repeated k times (including $k = 0$, where we omit the numbers between vertical bars):

$$\text{Pattern for } n = 4k + 8: \begin{array}{cccc|cccc|c} 5 & & & & 7 & & 6 & & 7 & & 4 \\ & 4 & & & & 5 & & 4 & & 5 & \end{array}$$

$$\text{Pattern for } n = 4k + 10: \begin{array}{cccc|cccc|c} 5 & & 5 & & 4 & & 6 & & 7 & & 4 \\ & 4 & & 7 & & 5 & & 4 & & 5 & \end{array}$$

$$\text{Pattern for } n = 6: \begin{array}{cc} 4 & 4 \\ & 5 \end{array}$$

Case 2. α is in 1-position, see Figure 4.

Consider the face ω and its descendants with respect to \mathcal{T}_G , and note that the color of their big vertices could already have been determined (with the exception of big vertices arising from α). Also, as the face α is in 1-position, there can be a big vertex arising from α , which is at distance 2 from $C(\omega)$. If this vertex exists, we denote it by p . By definition, p is a big vertex arising from ω , hence it is already colored. Also, since v is a big vertex arising from ω , it is also already colored.

By property (iii), the neighbors of v' , different from v , are not in B , and there can be at most one big vertex different from v at distance 4 from p , which is already colored. If such a vertex exists, we denote it by r . All other big vertices that are already colored are at distance at least 8 from the vertices arising from α .

Similarly as in Case 1 (considering the distance from v and using property (iii)), we find the position of big vertices that possibly arise from α , and depict

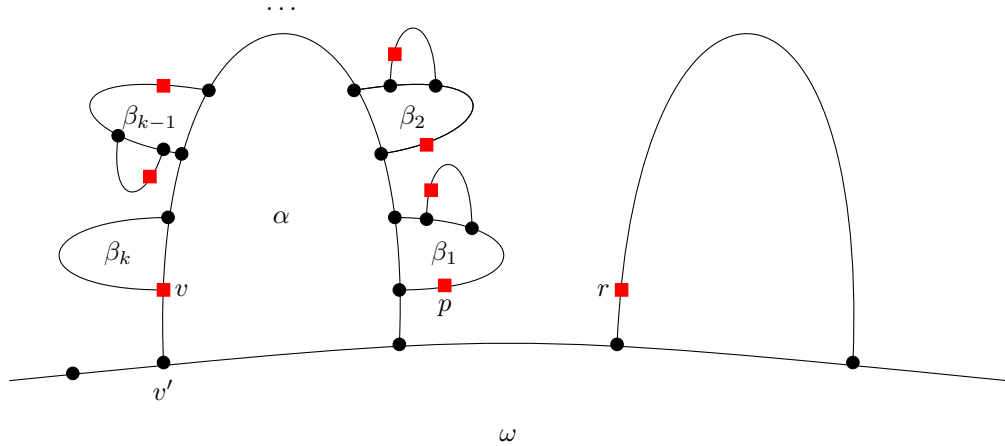


Figure 4: Big vertices arising from face α in 1-position (circle: vertex of A ; square: vertex of B , i.e. big vertex).

them in Figure 5. Note that some of the big vertices indicated in this figure may not exist in G .

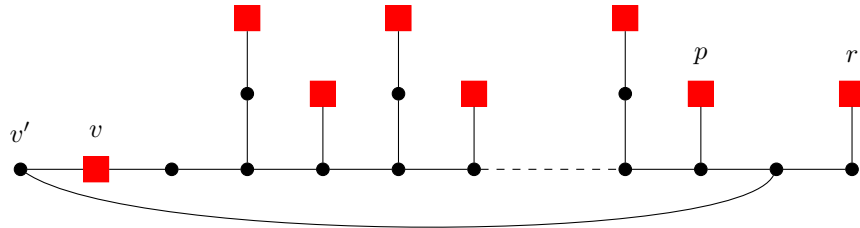


Figure 5: Case 2 (α is in 1-position).

Since, by property (iii), the vertices v , p and r (see Figure 5) are pairwise at distance 4 from each other, exactly one of them has a very big color. Hence, in a similar way as in Case 1, we only consider three subcases, depending on the colors of the vertices v , p and r . These three subcases give the way to deal with every possible color configuration for v , p and r . (Also, when r or p does not exist and no vertex among v , p and r is very big, it is possible to deal with this situation by considering that a non existing vertex among v , p and r is very big.) The various cases are described in the following table:

Subcase	a	b	c
$f(v)$	6	4	4
$f(p)$	4	6	5
$f(r)$	5	5	6

Note that if the length n of $C(\alpha)$ is 4, there are no big vertices arising from α , and this case is trivially resolved. Hence, we suppose that $n \geq 6$.

Subcase 2.a. $f(v) = 6$, $f(p) = 4$ and $f(r) = 5$. We present patterns that define the coloring of the big vertices with respect to different lengths n of the cycle $C(\alpha)$. The patterns give f -values of the big vertices in order of their presentation in Figure 5. As in Case 1, if some of the big vertices that arise from α do not exist, we simply skip the corresponding values in the pattern. Note that the numbers between vertical bars are to be repeated k times (case $k = 0$ included). Note also that the last value in the patterns represents $f(p) = 4$.

$$\text{Pattern for } n = 4k + 8: \quad 4 \quad 5 \quad \left| \quad 7 \quad 4 \quad 6 \quad 5 \quad \right| \quad 7 \quad 4$$

$$\text{Pattern for } n = 4k + 10: \quad 4 \quad 5 \quad \left| \quad 7 \quad 4 \quad 6 \quad 5 \quad \right| \quad 4 \quad 7 \quad 5 \quad 4$$

Note that because $f(v) = 6$, the first four and the last three values (including the value of p) in the patterns cannot be 6. In each of the patterns, one can check that property (iv) holds.

Cases $k = 0$, where the values between vertical bars are omitted, cover $n \in \{8, 10\}$, while the pattern for length $n = 6$ is: $\begin{array}{c} 5 \\ 4 \end{array}$

Subcase 2.b. $f(v) = 4$, $f(p) = 6$ and $f(r) = 5$. We present patterns that define the coloring of the big vertices with respect to different lengths n of the cycle $C(\alpha)$. Note that the numbers between vertical bars are to be repeated k times (case $k = 0$ included). The last value in the patterns represents $f(p) = 6$.

$$\text{Pattern for } n = 4k + 6: \quad \left| \quad 6 \quad 5 \quad 7 \quad 4 \quad \right| \quad 5 \quad 6$$

$$\text{Pattern for } n = 4k + 8: \quad 7 \quad 5 \quad \left| \quad 6 \quad 4 \quad 7 \quad 5 \quad \right| \quad 4 \quad 6$$

Note that $f(v) = 4$ implies that patterns must avoid having the first two values equal to 4. Cases $n \in \{6, 8\}$ are covered by the above patterns with $k = 0$ (i.e., removing the values between vertical bars).

Subcase 2.c. $f(v) = 4$, $f(p) = 5$ and $f(r) = 6$. The following two patterns define the coloring of the big vertices (the numbers between vertical bars are to be repeated k times, with $k = 0$ included, covering $n = 6$ and $n = 8$). The last value in the patterns represents $f(p) = 5$.

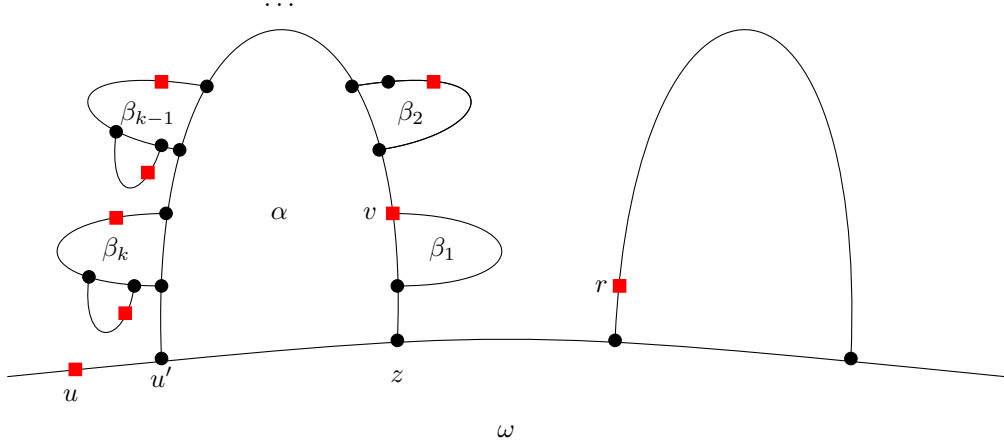


Figure 6: Big vertices arising from face α in 2-position (circle: vertex of A ; square: vertex of B , i.e. big vertex).

$$\text{Pattern for } n = 4k + 6: \left| \begin{array}{ccc} 7 & 6 & 7 \\ & 5 & 4 \end{array} \right| \begin{array}{c} 7 \\ 5 \end{array}$$

$$\text{Pattern for } n = 4k + 8: \left| \begin{array}{ccc} 7 & 6 & 5 \\ & 5 & 4 \end{array} \right| \begin{array}{cc} 5 & 4 \\ 7 & 5 \end{array}$$

In the above patterns, in particular, the first two values cannot be 4, as $f(v) = 4$, and the last three values cannot be 6, as $f(r) = 6$.

Case 3. α is in 2-position, see Figure 6.

Again we consider the face ω and its descendants with respect to \mathcal{T}_G , and note that the color of their big vertices could already have been determined (with the exception of big vertices arising from α). Also, by the definition of 2-position of the face α , u and v are already colored. There can be at most one additional vertex, which is already colored and is at distance 6 to a big vertex arising from α that is not yet colored; if this vertex exists, we denote it by r , see Figure 6.

Similarly as in Case 1 (considering the distance from v and using property (iii)), we find the possible positions of big vertices that arise from α , and depict them in Figure 7. Note that some of the big vertices indicated in this figure may not exist in G .

Since the vertices u , v and r (see Figures 6 and 7) are pairwise at distance 4 from each other, exactly one of them has a very big color. Hence, in a similar way as in Case 1, we only consider three subcases, depending on which vertex

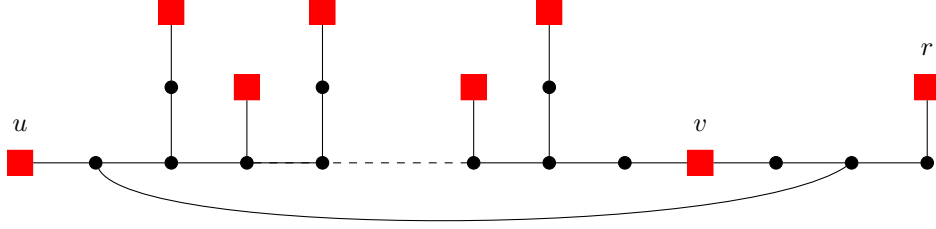


Figure 7: Case 3 (α is in 2-position).

among u , v and r is very big. These three subcases give the way to deal with every possible color configuration for u , v and r . (Also, in the case r does not exist and no vertex among u and v is very big, it is possible to deal with this situation by considering that r is very big.) The cases are described in the following table:

Subcase	a	b	c
$f(u)$	4	6	5
$f(v)$	6	4	4
$f(r)$	5	5	6

If $n = 4$ there are no big vertices arising from α . Hence, we suppose that $n \geq 6$.

Subcase 3.a. $f(u) = 4$, $f(v) = 6$ and $f(r) = 5$. The following two patterns define the coloring of the big vertices (the numbers between vertical bars are to be repeated k times, including $k = 0$).

$$\text{Pattern for } n = 4k + 8: \left| \begin{array}{cc} 7 & 6 \\ & 5 \end{array} \right| \left| \begin{array}{cc} 7 & 4 \\ & 5 \end{array} \right|$$

$$\text{Pattern for } n = 4k + 10: \left| \begin{array}{ccc} 7 & 6 & \\ & 5 & 4 \end{array} \right| \left| \begin{array}{ccc} 5 & 5 & 5 \\ & 7 & 4 \end{array} \right|$$

Note that because $f(v) = 6$ and $f(u) = 4$, the last four values in the patterns cannot be 6 and the first two values in the patterns cannot be 4 or 6. In each of the patterns, one can check that property (iv) holds.

For $n = 6$, there can be at most one big vertex arising from α , and we can color it by 5. For $n = 4$ there are no big vertices arising from α .

Subcase 3.b. $f(u) = 6$, $f(v) = 4$ and $f(r) = 5$. The following two patterns define the coloring of the big vertices (the numbers between vertical bars are to be repeated k times, including $k = 0$).

$$\text{Pattern for } n = 4k + 8: \left| \begin{array}{cc} 4 & \\ & 5 \end{array} \right| \left| \begin{array}{cc} 7 & 6 \\ & 4 \end{array} \right| \left| \begin{array}{cc} & \\ 5 & 7 \end{array} \right|$$

Pattern for $n = 4k + 10$: $5 \quad 7 \quad 5 \mid 6 \quad 7 \quad 5 \mid 6$
 $4 \quad 4 \quad 5$

Note that because $f(u) = 6$ and $f(v) = 4$, the first four values in the patterns cannot be 6 and the last two values in the patterns cannot be 4. In each of the patterns, one can check that property (iv) holds.

For $n = 6$, there can be at most one big vertex arising from α , and we can color it by 5. For $n = 4$ there are no big vertices arising from α .

Subcase 3.c. $f(u) = 5$, $f(v) = 4$ and $f(r) = 6$. Finally, the following two patterns define the coloring of the big vertices (the numbers between vertical bars are to be repeated k times, $k = 0$ included).

Pattern for $n = 4k + 6$: $7 \quad 6 \quad 7 \mid 7$
 $4 \quad 5$

Pattern for $n = 4k + 8$: $7 \quad 6 \quad 4 \quad 5 \mid 4 \quad 5$
 $4 \quad 7$

Note that because $f(r) = 6$, $f(v) = 4$ and $f(u) = 5$, the first two values in the patterns cannot be 5 or 6 and the last two values in the patterns cannot be 4. In each of the patterns, one can check that property (iv) holds.

It is straightforward to see that in all of the above patterns property (iv) is satisfied, and that the resulting coloring f is a packing coloring that uses 7 colors. This completes the proof. \square

We complete this section by showing that the upper bound of 7 on the packing chromatic number of 2-connected bipartite subcubic outerplanar graphs is sharp.

Proposition 5. *There exists a 2-connected bipartite subcubic outerplanar graph G such that $\chi_\rho(G) \geq 7$.*

Proof. Let \mathcal{T} be the infinite binary tree. Sloper [23] has proven that $\chi_\rho(\mathcal{T}) = 7$. A consequence is that there exists a finite subcubic tree T such that $\chi_\rho(T) = 7$. Let d be the depth of T . Finally, let $k = 2d + 2$.

Let x and y be two adjacent vertices of degree 2 in a graph G . Adding a k -cycle on x and y is an operation that consists in adding a path of $k - 2$ vertices to G and joining one endvertex of the path to x and the other endvertex to y . Let G_1 be a cycle of order k . Let G_{i+1} , $i \geq 1$, be the graph obtained from G_i by adding a k -cycle on every adjacent pair of vertices of degree 2 in G_i , where we arrange these pairs in such a way that each vertex of degree 2 belongs to one adjacent pair; this can be done by adding k -cycles on adjacent vertices of degree 2 following the outer cycle of G_i . In this way, in G_{i+1} there does not remain any vertex of degree 2 from $V(G_i)$. Note that, by construction, G_i is a 2-connected bipartite subcubic outerplanar graph for every integer $i \geq 1$.

Let u be a vertex that belongs to G_1 in the construction of G_k . Note that the set $\{v \in V(G_k) \mid d(u, v) \leq d\}$ induces a subcubic tree containing T as an induced subgraph. Thus, since every graph has a packing chromatic number larger than or equal to the packing chromatic number of any of its (induced) subgraphs, we derive $\chi_\rho(G_k) \geq 7$. \square

4 $(1, 3, \dots, 3)$ -packing coloring of bipartite outerplanar graphs

In this section, we need to extend the definition of \mathcal{T}_G in order to have an underlying tree even if the outerplanar graph is not 2-connected. For an outerplanar graph G , let $D = \{u \in V(G) \mid u \notin C(v), v \in V(\mathcal{T}_G)\}$ and let $A = V(G) \setminus D$. Note that the graph with vertex set $V(\mathcal{T}_G) \cup D$ and edge set $E(\mathcal{T}_G) \cup E(G[D])$ is a forest. We construct \mathcal{L}_G from the forest with vertex set $V(\mathcal{T}_G) \cup D$ and edge set $E(\mathcal{T}_G) \cup E(G[D])$ as follows. First, for each bridge uv of G such that $u \in A$ and $v \in D$, we add an edge to \mathcal{L}_G between v and an arbitrary face α containing u . Second, for each bridge uv of G such that $u \in A$ and $v \in A$, we add an edge to \mathcal{L}_G between an arbitrary face α containing u and an arbitrary face β containing v . Third, let G' be the graph obtained from G by removing the bridges. For a cut vertex u of G' , let B_1, \dots, B_k be the maximal 2-connected components of G' containing u and let $\alpha_i(u)$ be a face chosen arbitrarily among the faces from B_i containing u , $1 \leq i \leq k$. For each cut vertex u and each integer i between 2 and k , we add an edge to \mathcal{L}_G between $\alpha_1(u)$ and $\alpha_i(u)$.

It is easily seen that the graph \mathcal{L}_G is a tree for any outerplanar graph G . Figure 8 illustrates an example construction of the tree \mathcal{L}_G for an outerplanar graph G .

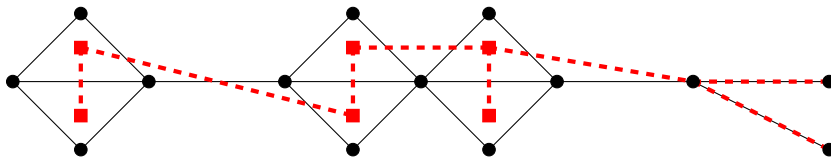


Figure 8: The graph \mathcal{L}_G for an outerplanar graph G (circle: vertex of G ; square: vertex of \mathcal{T}_G ; line: edge of G ; dashed line: edge of \mathcal{L}_G).

First, we prove Theorem 2, which states that a bipartite outerplanar graph with maximum degree Δ bounded by k is $(1, 3, \dots, 3)$ -colorable, where 3 appears k times in the sequence, $k \geq 3$.

Proof of Theorem 2.

In the construction of the S -packing coloring, the vertices with color 1 will form an independent set and the vertices with color a_i , $i \in \{1, \dots, k\}$ will form a 3-packing. The proof is by induction on the order of \mathcal{L}_G . If G has more than one vertex, vertices colored by 1 will correspond to one part of the bipartition of G .

For the induction, suppose that \mathcal{L}_G contains only one vertex u . If $u \in D$, then it suffices to color u with color 1. If $u \in V(\mathcal{T}_G)$, then we have to color the cycle $C(u)$. We can clearly color it with colors 1, a_1 , a_2 and a_3 by (repeatedly) using the pattern 1, a_1 , 1, a_2 , and using the color 1, a_3 for the last two vertices when the length of the cycle is not divisible by four.

Now, consider a graph G such that \mathcal{L}_G has order $n + 1$, $n \geq 1$. Let u be a leaf of \mathcal{L}_G and let G' be the following graph:

$$G' = \begin{cases} G - u & \text{if } u \in D \\ G - B(u) & \text{if } u \in V(\mathcal{T}_G); \end{cases}$$

where $B(u)$ is the set of the vertices of $V(G) \setminus D$, which belong to $C(u)$ but no other inner face of G . By induction, we can color the vertices of G' , since $\mathcal{L}_{G'}$ has order n , and it suffices to extend the coloring of G' to the uncolored vertices of G . Let v be the neighbor of u in $\mathcal{L}_{G'}$.

Case 1 $u \in D$.

Let v' denote the neighbor of u in $C(v)$ if $v \in \mathcal{T}_G$, and $v' = v$ if $v \in D$. If v' is not colored by 1, then we can color u by 1 and we are done. Otherwise, when v' is colored by color 1, then note that v' has at most $k - 1$ colored neighbors, whose neighbors are all colored by 1. Therefore, we can color u by a color a_i , which is not given to any vertices of $N_G(v')$, since other vertices with color a_i are at distance at least 4 from u .

Case 2 $u \in V(\mathcal{T}_G)$ and $v \in D$.

Let u' be the neighbor of v in $C(u)$. Firstly, if v is colored by 1, then, since $\Delta(G) \leq k$, at most $k - 1$ neighbors of v are colored in G , and we can color u' by a color a_i not used in other neighbors of v . We can color the remaining uncolored vertices of $C(u)$ by using the color 1 and three more colors (proceeding in the same way as in the coloring of a cycle described in the initial step of the induction). Secondly, if v is not colored by color 1, then we color u' with color 1. We can again extend the coloring to the remaining vertices of $C(u)$ using color 1 and three more colors in an analogous way as in the initial step of the induction. We just need to avoid that a neighbor of u' is assigned the same color as v , which is always possible since $k \geq 3$.

Case 3 $u \in V(\mathcal{T}_G)$ and $v \in V(\mathcal{T}_G)$.

Subcase 3.a $|V(C(u)) \cap V(C(v))| = 0$.

In this case, a vertex $u' \in C(u)$ is adjacent to a vertex $v' \in C(v)$. For the proof of this case, we follow the same steps as in Case 2, where the vertex v' , defined in the previous sentence, plays the role of v in Case 2.

Subcase 3.b $|V(C(u)) \cap V(C(v))| = 1$.

Let $\{w\} = V(C(u)) \cap V(C(v))$. Suppose w is colored by color 1. Since w is in the uncolored facial cycle $C(u)$, at most $k - 2$ neighbors of w are colored so far in G . Thus, we can give two (distinct) colors not used by neighbors of w to the two neighbors of w in $C(u)$ and easily extend the coloring to the remaining uncolored vertices of $C(u)$ using these two colors, color 1 and possibly a third color a_i (that will be given to a vertex at distance at least 4 from any vertex with color a_i of another cycle). Now, if w is not

colored with color 1, then every neighbor of w can be colored with color 1. It is again easy to color the remaining uncolored vertices of $C(u)$ by using color 1 and three more colors, applying the pattern as in the initial step of the induction.

Subcase 3.c $|V(C(u)) \cap V(C(v))| = 2$.

Let $\{w_1, w_2\} = V(C(u)) \cap V(C(v))$. Since G is outerplanar, w_1 and w_2 are adjacent in G . Without loss of generality suppose w_1 is colored by color 1. By the induction hypothesis w_2 is already colored. Let x be the neighbor of w_1 (which is not w_2) in $C(u)$. Since the vertex w_1 has at most $k - 1$ colored neighbors, we can color the vertex x by a color a_i that is not used in any of the neighbors of w_1 . The remaining vertices of $C(u)$ can be colored by using color 1 and three more colors (that of w , of x and a third one) in the same way as in the previous cases.

□

In the following proposition, we prove that Theorem 2 does not hold if in S an integer 3 is replaced by an integer 4.

Proposition 6. *There exists a bipartite outerplanar graph G with $\Delta(G) \leq k$, which is not S -packing colorable for the list $S = (1, 3, \dots, 3, 4)$ containing $k - 1$ times the integer 3.*

Proof. Let T be the complete k -ary tree of depth 5 and root r , and suppose there exists an S -coloring c of T , using $S = (1, 3, \dots, 3, 4)$ as in the statement of the proposition. Note that there exists a vertex $x \in \{r\} \cup N(r)$ that is colored by 1. Since x has k neighbors, all must receive distinct colors, different from 1. In particular, there exists a neighbor y of x such that $c(y) = 4$. Let z be any neighbor of x different from y . Clearly, z must be colored by a color '3', while all neighbors of z are colored by 1. Finally, the neighbors of the vertices in $N(z)$ must receive all colors different from 1. In particular, there exists a vertex u with $d(u, y) = 4$, such that $c(u) = 4$, which is a contradiction.

□

5 $(1, 2, 2, 2)$ -packing coloring of subcubic outerplanar graphs

In this section, we first prove that a subcubic outerplanar graph G is $(1, 2, 2, 2)$ -packing colorable when G is triangle-free outerplanar.

Proof of Theorem 3.

Let G be a subcubic outerplanar graph and let D and \mathcal{L}_G be defined as in Section 4. In this proof, the vertices with color 1 will form an independent set and the vertices with color a_i , for each $i \in \{1, 2, 3\}$ will form a 2-packing.

By induction on the order of \mathcal{L}_G , we prove that there is an S -packing coloring of G . Suppose \mathcal{L}_G has only one vertex u . If $u \in D$, then it suffices to color u

with color 1. If $u \in V(\mathcal{T}_G)$, then we have to color a cycle of order n . We color it with colors 1, a_1 , a_2 and a_3 using the pattern 1, a_1 , 1, a_2 and using the color a_3 for the last vertex if $n \equiv 1 \pmod{4}$, colors 1, a_3 for the two last vertices if $n \equiv 2 \pmod{4}$ or colors 1, a_3, a_2 for the three last vertices if $n \equiv 3 \pmod{4}$.

Now, consider a graph G such that \mathcal{L}_G has order $n + 1$, $n \geq 1$. Let u be a leaf of \mathcal{L}_G and let G' be the following graph:

$$G' = \begin{cases} G - u & \text{if } u \in D \\ G - B(u) & \text{if } u \in V(\mathcal{T}_G); \end{cases}$$

where $B(u)$ is the set of the vertices of $V(G) \setminus D$, which belong to $C(u)$ but no other inner face of G . By induction, we can color the vertices of G' , since $\mathcal{L}_{G'}$ has order n , and it suffices to extend the coloring of G' to the uncolored vertices of G . Let v be the neighbor of u in $\mathcal{L}_{G'}$.

Case 1 $u \in D$.

Let v' denote the neighbor of u in $C(v)$ if $v \in \mathcal{T}_G$ and $v' = v$ if $v \in D$. If v' is not colored by 1, then we can color u by 1 and we are done. Otherwise (v' is colored by 1), since G is subcubic, v' has at most two colored neighbors. Therefore, we can color u by a color not given to vertices from $N_G(v')$.

Case 2 $u \in \mathcal{T}_G$ and $v \in D$.

Let u' be the neighbor of v in $C(u)$. First, if v is colored by color 1, then, since $\Delta(G) \leq 3$, at most two neighbors of v are colored in G and we can easily color u' . We can color the remaining uncolored vertices of $C(u)$ by the color 1 and the three remaining colors (by proceeding as in the coloring of a cycle described in the initial step of the induction). Second, if v is not colored by color 1, then we color u' with color 1. Since u' is in $C(u)$ which is uncolored, exactly one neighbor of u' , namely v , is colored in G . Thus, we can give two different colors from $\{a_1, a_2, a_3\}$ to the neighbors of u' in $C(u)$. We now extend the coloring to the remaining vertices of $C(u)$ using color 1 and the three remaining colors (by proceeding as in the coloring of a cycle described in the initial step of the induction).

Case 3 $u \in \mathcal{T}_G$ and $v \in \mathcal{T}_G$.

Since G is subcubic, $|V(C(u)) \cap V(C(v))| = 0$ or $|V(C(u)) \cap V(C(v))| = 2$ (if $|V(C(u)) \cap V(C(v))| = 1$, then the common vertex would have degree at least 4).

Subcase 3.a $|V(C(u)) \cap V(C(v))| = 0$.

In this case, a vertex $u' \in C(u)$ is adjacent to a vertex $v' \in C(v)$. For the proof of this case, we follow the same steps as in Case 2, where the vertex v' , defined in the previous sentence, plays the role of v in Case 2.

Subcase 3.b $|V(C(u)) \cap V(C(v))| = 2$.

Let $\{w_1, w_2\} = V(C(u)) \cap V(C(v))$. Since G is outerplanar, w_1 and w_2 are adjacent in G . By the induction hypothesis, w_1 and w_2 are already

colored. Let x_1 be the neighbor of w_1 (which is not w_2) in $C(u)$ and let x_2 be the neighbor of w_2 (which is not w_1) in $C(u)$. If w_1 has no neighbor with color 1, then we recolor it with color 1. If after this, w_2 has no neighbor with color 1, then we recolor it with color 1.

Suppose that $4 \leq |C(u)| \leq 5$. First, if one of w_i is colored by 1, say w_1 , we can extend the coloring to $C(u)$ as follows. In this case, x_1 receives the color a_i , which is not used in the neighborhood of w_1 , and we can color x_2 by 1. In the case $|C(u)| = 5$, the common neighbor of x_1 and x_2 receives the color that was given to the neighbor of w_1 , which is not w_2 , in $C(v)$. Second, assume without loss of generality that w_i received color a_i for $i \in \{1, 2\}$. Note that by the above recoloring condition, w_1 and w_2 have neighbors, which are given color 1. If $|C(u)| = 4$, we can color x_1 and x_2 by colors 1 and a_3 , respectively. Otherwise, if $|C(u)| = 5$, we give color 1 to vertices x_1 and x_2 , and the common neighbor of x_1 and x_2 gets color a_3 .

So, let $|C(u)| > 5$. We color the uncolored vertices of $C(u)$ starting by coloring the vertex x_1 . We color x_1 by using a color not given to w_1 and the colored neighbors of w_1 . If w_2 is not colored by 1 we color x_2 by 1. Otherwise, we color x_2 by a color not given to w_2 and the colored neighbors of w_2 . The remaining vertices of $C(u)$ can be colored by the pattern described in the initial step of the induction, alternating 1, a_i , 1, a_j , and possibly completing the coloring of the cycle with the third color a_k , depending on the length of $C(u)$.

□

We next present two examples, which show that Theorem 3 is best possible. First, we prove that the result does not hold if the graph contains triangles.

Proposition 7. *There exists a subcubic outerplanar graph, which is not $(1, 2, 2, 2)$ -packing colorable.*

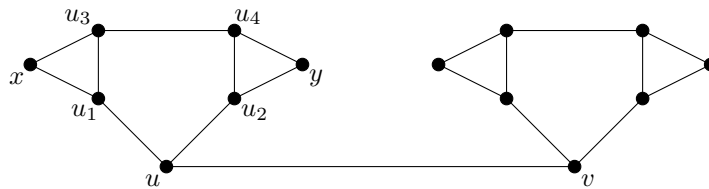


Figure 9: A non $(1, 2, 2, 2)$ -packing colorable graph.

Proof. Let G be the graph depicted in Figure 9. We suppose that a $(1, 2, 2, 2)$ -packing coloring uses the colors 1, a_1, a_2 and a_3 , the meaning of which should be clear. Suppose, on the contrary, that G has a $(1, 2, 2, 2)$ -packing coloring.

Clearly, one vertex among u and v should be colored by a color different from 1. Suppose, without loss of generality that u has a color in $\{a_1, a_2, a_3\}$, say a_1 . Note that at least one of the vertices u_3 and u_4 is not colored by color 1, and assume without loss of generality that u_3 is colored with a_2 . Consequently, we can only use colors 1 and a_3 for the three vertices u_2, u_4, y (forming a triangle), which is not possible. \square

Second, we prove that Theorem 3 cannot be improved by replacing in $(1, 2, 2, 2)$ an integer 2 by an integer 3, when G is in the class of subcubic triangle-free outerplanar graphs.

Proposition 8. *There exists a subcubic triangle-free outerplanar graph, which is not $(1, 2, 2, 3)$ -packing colorable.*

Proof. Let G_{25} be the graph obtained from six copies of the 5-cycle, one of which we distinguish and denote by C ; and to each vertex x of C we add an edge between x and a vertex of its own copy of C_5 . Note that in a $(1, 2, 2, 3)$ -packing coloring of C_5 one must color two vertices by color 1, and each of the other three vertices receives its own color among $\{2, 2', 3\}$. Each vertex x of the central cycle C is at distance at most 3 from all vertices in the 5-cycle, which is attached to x . Thus, the assumption that G_{25} is $(1, 2, 2, 3)$ -packing colorable implies that a vertex x in C is colored by color 3. However, distances from x to vertices of the 5-cycle attached to x prevent the use of color 3 in that copy of C_5 , which is in contradiction to the existence of a $(1, 2, 2, 3)$ -packing coloring of G_{25} . \square

6 Concluding remarks

Theorem 1 gives a partial (affirmative) answer to the question posed in several papers concerning the boundedness of the packing chromatic number in the class of planar subcubic graphs. Instead of repeating the question, we propose two problems that lie between Theorem 1 and this question. In one of them, we consider a non-bipartite extension of the theorem, and in the other we replace outerplanar graphs with planar graphs.

Question 1. *Is the packing chromatic number bounded in the class of 2-connected subcubic outerplanar graphs?*

Question 2. *Is the packing chromatic number bounded in the class of 2-connected bipartite subcubic planar graphs?*

While we do not dare to suggest an answer to the above questions, we strongly believe that Theorem 1 could be extended from the 2-connected case to all bipartite subcubic outerplanar graphs.

In Section 5, we proved that a subcubic triangle-free outerplanar graph is $(1, 2, 2, 2)$ -packing colorable, and it is not $(1, 2, 2, 3)$ -packing colorable in general. A similar proof as in Theorem 3 can be used to prove the following result.

Theorem 9. *If G is a subcubic triangle-free outerplanar graph, then G is $(1, 1, 2)$ -packing colorable.*

The above result can be viewed as an extension of the 3-colorability of outerplanar graphs. We cannot omit the triangle-free condition from Theorem 9, as demonstrated by the following example. Take four copies of the triangle C_3 , one of which we distinguish and denote it by C ; and to each vertex x of C we add an edge between x and a vertex of its own copy of C_3 . It is easy to see that the resulting graph G is not $(1, 1, 2)$ -packing colorable (clearly, G is outerplanar and subcubic). In addition, Theorem 9 cannot be improved in such a way that the integer 2 be replaced by 3 in $(1, 1, 2)$ -packing colorability of subcubic triangle-free outerplanar graphs. To see this, take the graph G_{25} from the proof of Proposition 8 as an example.

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References

- [1] G. Agnarsson, M. Halldórsson, Vertex coloring the square of outerplanar graphs of low degree, *Discuss. Math. Graph Theory* 30 (2010) 619–636.
- [2] G. Argiroffo, G. Nasini, P. Torres, The packing coloring problem for lobsters and partner limited graphs, *Discrete Appl. Math.* 164 (2014) 373–382.
- [3] J. Balogh, A. Kostochka, X. Liu, Packing chromatic number of subcubic graphs, *Discrete Math.* 341 (2018) 474–483.
- [4] J. Balogh, A. Kostochka, X. Liu, Packing chromatic number of subdivisions of cubic graphs, *Graphs Combin.* 35 (2019) 513–537 (2019).
- [5] M. Barnaby, F. Raimondi, T. Chen, J. Martin, The packing chromatic number of the infinite square lattice is between 13 and 15, *Discrete Appl. Math.* 225 (2017) 136–142.
- [6] B. Brešar, J. Ferme, An infinite family of subcubic graphs with unbounded packing chromatic number, *Discrete Math.* 341 (2018) 2337–2342.

- [7] B. Brešar, S. Klavžar, D.F. Rall, On the packing chromatic number of Cartesian products, hexagonal lattice, and trees, *Discrete Appl. Math.* 155 (2007) 2303–2311.
- [8] B. Brešar, S. Klavžar, D.F. Rall, Packing chromatic number of base-3 Sierpiński graphs, *Graphs Combin.* 32 (2016) 1313–1327.
- [9] B. Brešar, S. Klavžar, D.F. Rall, K. Wash, Packing chromatic number under local changes in a graph, *Discrete Math.* 340 (2017) 1110–1115.
- [10] B. Brešar, S. Klavžar, D.F. Rall, K. Wash, Packing chromatic number, $(1, 1, 2, 2)$ -colorings, and characterizing the Petersen graph, *Aequationes Math.* 91 (2017) 169–184.
- [11] J. Ekstein, P. Holub, O. Togni, The packing coloring of distance graphs $D(k, t)$, *Discrete Appl. Math.* 167 (2014) 100–106.
- [12] J. Fiala, P.A. Golovach, Complexity of the packing coloring problem for trees, *Discrete Appl. Math.* 158 (2010) 771–778.
- [13] J. Fiala, S. Klavžar, B. Lidický, The packing chromatic number of infinite product graphs, *European J. Combin.* 30 (2009) 1101–1113.
- [14] A. Finbow, D.F. Rall, On the packing chromatic number of some lattices, *Discrete Appl. Math.* 158 (2010) 1224–1228.
- [15] N. Gastineau, O. Togni, S -packing colorings of cubic graphs, *Discrete Math.* 339 (2016) 2461–2470.
- [16] N. Gastineau, P. Holub, O. Togni, On packing chromatic number of sub-cubic outerplanar graphs, *Discrete Appl. Math.* 255 (2019) 209–221.
- [17] W. Goddard, S.M. Hedetniemi, S.T. Hedetniemi, J.M. Harris, D.F. Rall, Broadcast chromatic numbers of graphs, *Ars Combin.* 86 (2008) 33–49.
- [18] Y. Jacobs, E. Jonck, E.J. Joubert, A lower bound for the packing chromatic number of the Cartesian product of cycles, *Cent. Eur. J. Math.* 11 (2013) 1344–1357.
- [19] D. Korže, A. Vesel, On the packing chromatic number of square and hexagonal lattice, *Ars Math. Contemp.* 7 (2014) 13–22.
- [20] D. Laïche, I. Bouchemakh, E. Sopena, On the packing coloring of undirected and oriented generalized theta graphs, *Australas. J. Combin.* 66 (2016) 310–329.
- [21] K-W. Lih, W-F. Wang, Coloring the square of an outerplanar graph, *Taiwanese J. Math.* 10 (2006) 1015–1023.
- [22] Z. Shao, A. Vesel, Modeling the packing coloring problem of graphs, *Appl. Math. Model.* 39 (2015) 3588–3595.

- [23] C. Sloper, An eccentric coloring of trees, *Austral. J. Combin.* 29 (2004) 309–321.
- [24] R. Soukal, P. Holub. A note on packing chromatic number of the square lattice, *Electron. J. Combin.* 17 (2010) 447–468.
- [25] O. Togni, On packing colorings of distance graphs, *Discrete Appl. Math.* 167 (2014) 280–289.
- [26] P. Torres, M. Valencia-Pabon, The packing chromatic number of hypercubes, *Discrete Appl. Math.* 190–191 (2015) 127–140.