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Casoratian solutions to the functional difference Toda equation

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Abstract

By using a particular Darboux transformation which we can call centered Darboux transformation, we construct solutions of the functional difference Toda lattice equation in terms of Casorati determinants. We give a complete description of the method and the corresponding proofs. We construct some explicit solutions for the first orders.

1 Introduction

The Toda lattice equation was first introduced in 1967 in [84] in which some exact solutions were constructed. In particular, it was integrated for initial conditions rapidly decreasing at infinity by Manakov [77] and Flaschka [4].

This equation can be represented in different normalizations.

We consider the functional difference Toda equation which can be written as

$$\partial_t^2 g(x, t) = \exp[g(x - h, t) - g(x, t)] - \exp[g(x, t) - g(x + h, t)]. \quad (1)$$

Writing this equation as a compatibility condition of a system of two equations, we use a special type of Darboux transformation defined by the following formula

$$D_1(f)(x, t) = \psi_1(f)(x, t) = f(x - \frac{h}{2}, t) - \frac{f_1(x - \frac{h}{2}, t)}{f_1(x + \frac{h}{2}, t)} f(x + \frac{h}{2}, t), \quad (2)$$

f and f_1 being two independent solutions of the system of compatibility of the Toda equation, associated with two different spectral parameters.

By using this, we construct explicit solutions of this equation for the first orders. Some results about this equation were given by Date and Tanaka [2, 3], but the expressions of these solutions depend on integrals of Riemann theta function and don't give any explicit or easily calculable expression, contrary to the present method.

The construction of the multisoliton solutions for the Toda lattice using a binary Darboux transformation, different from the one given in this text, was studied in the paper [1]. But, still in this paper, the solutions of the Toda equation were

given in terms of Bessel and Neumann functions, which makes these expressions difficult to use. In particular, explicit solutions are difficult to calculate.

2 Functional difference Toda equation

2.1 Different representations of this equation

We consider the functional difference Toda equation (1) which can be rewritten as :

$$g_{tt}(x, t) = \exp[g(x - h, t) - g(x, t)] - \exp[g(x, t) - g(x + h, t)], \quad (3)$$

where as usual, the subscripts represent partial derivatives.

This equation was deeply studied, but it is only in 1974 that its integrability has been proven by Manakov and Flaschka [77, 4] in the case $x = n$, $h = 1$.

Mikhailov constructed the Lax pair for the Toda lattice equation [82].

This Toda equation can be seen as a compatibility condition of (4) and (5). It is the representation of the functional difference of Toda lattice equation(3) :

$$v(x, t)f(x + h, t) + f(x - h, t) + b(x, t)f(x, t) = \lambda f(x, t), \quad (4)$$

$$f_t(x, t) = b(x, t)f(x, t) + f(x - h, t). \quad (5)$$

Proposition 2.1 *The compatibility conditions of equations (4) and (5) are equivalent to equation (3).*

We can note that the set of solutions of (4, 5) has infinite dimension because for every function h -periodic g (i.e. $g(x + h) = g(x)$), $g(x)f(x)$ is still a solution of (4, 5) respectively. That makes the difference with the case $x = n$, $h = 1$ where for a fixed value of the spectral parameter, the set of the solutions to the equation (4) is 2-dimensional.

We can use the previous system to construct solutions by means of the Darboux transformation which will be defined below.

2.2 Covariance property and Darboux transformations

Among the different possibilities of Darboux transformations and the results of covariance, we had to choose the one that could give simple results for its iteration. We could have chosen

$$\begin{aligned} \psi_1(x, t) &= f(x, t) - \sigma_1(x, t)f(x + h, t) \\ \text{with } \sigma_1(x, t) &= \frac{f_1(x, t)}{f_1(x + h, t)}, \end{aligned} \quad (6)$$

or

$$\begin{aligned} \psi_1(x, t) &= f(x - h, t) - \sigma_1(x, t)f(x, t) \\ \text{with } \sigma_1(x, t) &= \frac{f_1(x - h, t)}{f_1(x, t)}, \end{aligned} \quad (7)$$

or

$$\begin{aligned}\psi_1(x, t) &= f(x + h, t) - \sigma_1(x, t)f(x, t) \\ \text{with } \sigma_1(x, t) &= \frac{f_1(x + h, t)}{f_1(x, t)},\end{aligned}\tag{8}$$

or more.

But the transformation that gave the simplest results for the coefficients in the iteration was the following, called centered Darboux transformation.

For an arbitrary solution $f(x, t)$ of (4, 5) generated by the fixed solution $f_1(x, t)$ of the same equations with $\lambda = \lambda_1$, we define the centered Darboux transformation $\psi_1(x, t)$ of the function f by the formula

$$\begin{aligned}\psi_1(x, t) &= f(x - \frac{h}{2}, t) - \sigma_1(x, t)f(x + \frac{h}{2}, t) = \frac{\begin{vmatrix} f(x - \frac{h}{2}, t) & f_1(x - \frac{h}{2}, t) \\ f(x + \frac{h}{2}, t) & f_1(x + \frac{h}{2}, t) \end{vmatrix}}{f_1(x + \frac{h}{2}, t)}, \\ \text{with } \sigma_1(x, t) &= \frac{f_1(x - \frac{h}{2}, t)}{f_1(x + \frac{h}{2}, t)},\end{aligned}\tag{9}$$

In the following, to avoid any confusion with the subscript, we will use the classical notation for the partial derivative ∂_t .

Theorem 2.1 *The function ψ_1 defined by*

$$\psi_1(x, t) = f(x - \frac{h}{2}, t) - \frac{f_1(x - \frac{h}{2}, t)}{f_1(x + \frac{h}{2}, t)}f(x + \frac{h}{2}, t),\tag{10}$$

$$\tag{11}$$

represents a general solution of the following equations

$$v_1(x, t)\psi_1(x + h, t) + \psi_1(x - h, t) + b_1(x, t)\psi_1(x, t) = \lambda\psi_1(x, t)\tag{12}$$

$$\partial_t\psi_1(x, t) = b(x, t)\psi_1(x, t) + \psi_1(x - h, t)$$

with

$$\begin{aligned}v_1(x - \frac{h}{2}, t) &= \frac{f_1(x + h, t)f_1(x - h, t)}{(f_1(x, t))^2}v(x, t), \\ b_1(x, t) &= b(x + \frac{h}{2}, t) + \partial_t \ln \left(\frac{f_1(x - \frac{h}{2}, t)}{f_1(x + \frac{h}{2}, t)} \right).\end{aligned}\tag{13}$$

The function ϕ_1 defined by

$$\phi_1(x, t) = \phi(x + \frac{h}{2}, t) + \ln \left(\frac{f_1(x - \frac{h}{2}, t)}{f_1(x + \frac{h}{2}, t)} \right),\tag{14}$$

is a solution of the Toda equation (3)

$$g_{tt}(x, t) = \exp[g(x - h, t) - g(x, t)] - \exp[g(x, t) - g(x + h, t)],$$

$\phi(x, t)$ being an initial solution to (3).

So, Darboux transformation maps (4, 5) into equations of the same forms, with the same value of spectral parameter λ , but with new potentials constructed in terms of the initial potentials $v(x, t)$, $b(x, t)$ and a fixed solution $f_1(x, t)$ of (4, 5).

2.3 Multiple Darboux transformation

We denote $d_m(x, t)$ the following determinant

$$d_m(x, t) = d_m(f_1, f_2, \dots, f_m)(x, t) = \det \left(f_k(x - (m + 1 - 2j)\frac{h}{2}, t) \right)_{j,k=1, \dots, m}. \quad (15)$$

The determinant (15) is a Casorati determinant.

Then we have the following statement which gives very simple expressions of $b_{m,t}$ and $v_{m,t}$ in terms of Casorati determinants :

Theorem 2.2 *The function ψ_m defined by*

$$\psi_m(f)(x, t) = \frac{d_{m+1}(f, f_1, \dots, f_m)(x, t)}{d_m(f_1, \dots, f_m)(x + \frac{h}{2}, t)}, \quad (16)$$

is a solution of,

$$v_m(x, t)\psi_m(f)(x + h, t) + \psi_m(f)(x - h, t) + b_m(x, t)\psi_m(f)(x, t) = \lambda\psi_m(f)(x, t), \quad (17)$$

$$\partial_t \psi_m(f)(x, t) = b(x, t)\psi_m(f)(x, t) + \psi_m(f)(x - h, t) \quad (18)$$

for $m \geq 1$,

$$v_m(x, t) = \frac{d_m(x - \frac{h}{2}, t)d_m(x + 3\frac{h}{2}, t)}{d_m(x + \frac{h}{2}, t)^2}v(x + m\frac{h}{2}, t), \quad (19)$$

$$b_m(x, t) = b(x + m\frac{h}{2}, t) + \partial_t \ln \left(\frac{d_m(x - \frac{h}{2})}{d_m(x + \frac{h}{2})} \right). \quad (20)$$

The function ϕ_m defined by

$$\phi_m(x, t) = \phi(x + m\frac{h}{2}, t) + \ln \left(\frac{d_m(x - \frac{h}{2})}{d_m(x + \frac{h}{2})} \right), \quad (21)$$

is a solution of the Toda equation (3)

$$g_{tt}(x, t) = \exp[g(x - h, t) - g(x, t)] - \exp[g(x, t) - g(x + h, t)],$$

$\phi(x, t)$ being an initial solution to (3).

2.4 Another formulation of the covariance property for the multiple Darboux transformation

In the previous theorem, the expression of b_m needed the presence of the derivation with respect to t and so a complex sum of $2m$ terms. We can give another simpler formulation for b_m which needs d_m and another determinant, $c_m(x, t)$ defined by

$$\begin{aligned} (c_m(x, t))_{1k} &= f_k(x + (m-1)\frac{h}{2}, t), \quad k = 1, \dots, m \\ (c_m(x, t))_{jk} &= f_k(x - (m-2j-1)\frac{h}{2}, t), \quad j = 2, \dots, m, \quad k = 1, \dots, m. \end{aligned} \quad (22)$$

The derivation of the determinant d_m can be written as

$$\partial_t d_m(x - \frac{h}{2}, t) = \sum_{i=1}^m d^i(x, t),$$

with

$$d^i(x, t) = \det(r_1, r_2, \dots, r_{i-1}, \partial_t r_i, r_{i+1}, \dots, r_m),$$

where r_j is a j -th row of $d_m(x - \frac{h}{2}, t)$.

By using (5) we can reduce $d^i(x, t)$ to

$$b(x - (m+1-2i)\frac{h}{2}, t) d_m(x, t),$$

for $i \geq 2$.

For $i = 1$, $d^1(x, t)$ can be written as

$$b(x - (m-1)\frac{h}{2}, t) d_m(x, t) + c_m(x - h, t).$$

The derivative

$$\partial_t d_m(x - \frac{h}{2}, t)$$

is thus reduced to

$$\sum_{k=1}^m b(x - (m+2-2k)\frac{h}{2}, t) d_m(x - \frac{h}{2}, t) + c_m(x - 3\frac{h}{2}, t).$$

Therefore the expression

$$b_m(x, t) = b(x + m\frac{h}{2}, t) + \partial_t \ln \frac{d_m(x - \frac{h}{2}, t)}{d_m(x + \frac{h}{2}, t)} = b(x + m\frac{h}{2}, t) + \frac{\partial_t d_m(x - \frac{h}{2}, t)}{d_m(x - \frac{h}{2}, t)} - \frac{\partial_t d_m(x + \frac{h}{2}, t)}{d_m(x + \frac{h}{2}, t)}$$

can be written

$$b_m(x, t) = b(x + m\frac{h}{2}, t) + \frac{\sum_{k=1}^m b(x - (m+2-2k)\frac{h}{2}, t) d_m(x - \frac{h}{2}, t) + c_m(x - 3\frac{h}{2}, t)}{d_m(x - \frac{h}{2}, t)}$$

$$-\frac{\sum_{k=1}^m b(x - (m - 2k)\frac{h}{2}, t)d_m(x + \frac{h}{2}, t) + c_m(x - \frac{h}{2}, t)}{d_m(x + \frac{h}{2}, t)}.$$

It can be rewritten in the form

$$b_m(x, t) = b(x + m\frac{h}{2}, t) + \sum_{k=0}^{m-1} b(x - (m - 2k)\frac{h}{2}, t) + \frac{c_m(x - 3\frac{h}{2}, t)}{d_m(x - \frac{h}{2}, t)} - \sum_{k=1}^m b(x - (m - 2k)\frac{h}{2}, t) - \frac{c_m(x - \frac{h}{2}, t)}{d_m(x + \frac{h}{2}, t)}.$$

The previous expression can be simplified, and we obtain,

$$b_m(x, t) = b(x - m\frac{h}{2}, t) + \frac{c_m(x - 3\frac{h}{2}, t)}{d_m(x - \frac{h}{2}, t)} - \frac{c_m(x - \frac{h}{2}, t)}{d_m(x + \frac{h}{2}, t)},$$

So we get the following statement

Theorem 2.3

$$\psi_m(f)(x, t) = \frac{d_{m+1}(f, f_1, \dots, f_m)(x, t)}{d_m(f_1, \dots, f_m)(x + \frac{h}{2}, t)}, \quad (23)$$

is a solution of,

$$v_m(x, t)\psi_m(f)(x + h, t) + \psi_m(f)(x - h, t) + b_m(x, t)\psi_m(f)(x, t) = \lambda\psi_m(f)(x, t), \quad (24)$$

$$\partial_t \psi_m(f)(x, t) = b_m(x, t)\psi_m(f)(x, t) + \psi_m(f)(x - h, t) \quad (25)$$

for $m \geq 1$,

$$v_m(x, t) = \frac{d_m(x - \frac{h}{2}, t)d_m(x + 3\frac{h}{2}, t)}{d_m(x + \frac{h}{2}, t)^2}v(x + m\frac{h}{2}, t), \quad (26)$$

$$b_m(x, t) = b(x - m\frac{h}{2}, t) + \frac{c_m(x - 3\frac{h}{2}, t)}{d_m(x - \frac{h}{2}, t)} - \frac{c_m(x - \frac{h}{2}, t)}{d_m(x + \frac{h}{2}, t)}, \quad (27)$$

2.5 Explicit solutions

In the following, we construct explicit solutions to the functional difference Toda equation. We can compare these solutions with those given in the discrete case, but very limited, for example, in articles [83, 78, 79].

2.5.1 Case of order 1

Example 2.1 The function f_1 defined by

$$f_1(x, t) = e^{t \cos a_1 h} \sin(a_1 x - t \sin(a_1 h) + c_1) \quad (28)$$

is a solution of the system (4–5) with $v_1 = 1$, $b_1 = 0$ and $\lambda_1 = 2 \cos(a_1 h)$.
The function ϕ_1 defined by

$$\begin{aligned} \phi_1(x, t) = & \ln(-e^{t \cos(a_1 h)} \sin(-a_1(x - 1/2 h) + t \sin(a_1 h) - c_1)) \\ & - \ln(-e^{t \cos(a_1 h)} \sin(-a_1(x + 1/2 h) + t \sin(a_1 h) - c_1)) \end{aligned} \quad (29)$$

is a solution of the Toda equation (3)

$$g_{tt}(x, t) = \exp[g(x - h, t) - g(x, t)] - \exp[g(x, t) - g(x + h, t)],$$

In the following, we choose particular generating functions to construct using the Darboux transformation, explicit solutions to the Toda equation.

Choosing the functions f_j , solutions of the system (4-5) for elementary potentials $v(x, t) = 1$ and $b(x, t) = 0$. We can choose $\phi(x, t) = 0$ as initial solution to the Toda equation (3).

We can express the following result :

Theorem 2.4 Let f_j be the functions defined by

$$f_j(x, t) = e^{t \cos a_j h} \sin(a_j x - t \sin(a_j h) + c_j), \quad \lambda_j = 2 \cos(a_j h) \quad (30)$$

Then the function ψ_m defined by

$$\psi_m(x, t) = \frac{d_{m+1}(f_{m+1}, f_1, \dots, f_m)(x)}{d_m(f_1, \dots, f_m)(x + \frac{h}{2})} \quad (31)$$

is a solution of,

$$v_m(x, t)\psi_m(x + h, t) + \psi_m(x - h, t) + b_m(x, t)\psi_m(x, t) = \lambda_{m+1}\psi_m(x, t), \quad (32)$$

$$\partial_t \psi_m(x, t) = b_m(x, t)\psi_m(x, t) + \psi_m(x - h, t) \quad (33)$$

for $m \geq 1$,

$$v_m(x, t) = \frac{d_m(x - \frac{h}{2}, t)d_m(x + \frac{3h}{2}, t)}{d_m(x + \frac{h}{2}, t)^2}, \quad (34)$$

$$b_m(x, t) = \frac{c_m(x - \frac{3h}{2}, t)}{d_m(x - \frac{h}{2}, t)} - \frac{c_m(x - \frac{h}{2}, t)}{d_m(x + \frac{h}{2}, t)}. \quad (35)$$

Then the function ϕ_m defined by

$$\phi_m(x, t) = \ln \left(\frac{d_m(x - \frac{h}{2}, t)}{d_m(x + \frac{h}{2}, t)} \right), \quad (36)$$

is a solution of the Toda equation (3)

$$g_{tt}(x, t) = \exp[g(x - h, t) - g(x, t)] - \exp[g(x, t) - g(x + h, t)]. \quad (37)$$

Proof : The functions f_j verify relations (4) and (5). So applying the multiple Darboux covariance property, we get this result.

□

In the following, we only give the expressions of the solutions of the Toda equation without giving the expressions of ψ_m , v_m , b_m .

2.5.2 Case of order 2

Example 2.2 The function ϕ_2 defined by

$$\phi_2(x, t) = \ln(n_{\phi_2}) - \ln(d_{\phi_2}), \quad (38)$$

$$n_{\phi_2} = -e^{t \cos(a_2 h)} \sin(-a_2(x-1/2 h) + 1/2 a_2 h + t \sin(a_2 h) - c_2) e^{t \cos(a_1 h)} \sin(a_1(x-1/2 h) + 1/2 a_1 h - t \sin(a_1 h) + c_1) + e^{t \cos(a_1 h)} \sin(-a_1(x-1/2 h) + 1/2 a_1 h + t \sin(a_1 h) - c_1) e^{t \cos(a_2 h)} \sin(a_2(x-1/2 h) + 1/2 a_2 h - t \sin(a_2 h) + c_2)$$

$$d_{\phi_2} = -e^{t \cos(a_2 h)} \sin(-a_2(x+1/2 h) + 1/2 a_2 h + t \sin(a_2 h) - c_2) e^{t \cos(a_1 h)} \sin(a_1(x+1/2 h) + 1/2 a_1 h - t \sin(a_1 h) + c_1) + e^{t \cos(a_1 h)} \sin(-a_1(x+1/2 h) + 1/2 a_1 h + t \sin(a_1 h) - c_1) e^{t \cos(a_2 h)} \sin(a_2(x+1/2 h) + 1/2 a_2 h - t \sin(a_2 h) + c_2)$$

is a solution of the Toda equation (3)

$$g_{tt}(x, t) = \exp[g(x-h, t) - g(x, t)] - \exp[g(x, t) - g(x+h, t)],$$

2.5.3 Case of order 3

Example 2.3 The function ϕ_3 defined by

$$\phi_3(x, t) = \ln(n_{\phi_3}(x, t)) - \ln(d_{\phi_3}(x, t))$$

is a solution to the Toda equation (3) with

$$n_{\phi_3}(x, t) = e^{t \cos(a_3 h)} \sin(-a_3(x-1/2 h) + a_3 h + t \sin(a_3 h) - c_3) e^{t \cos(a_2 h)} \sin(-a_2(x-1/2 h) + t \sin(a_2 h) - c_2) e^{t \cos(a_1 h)} \sin(a_1(x-1/2 h) + a_1 h - t \sin(a_1 h) + c_1) - e^{t \cos(a_3 h)} \sin(-a_3(x-1/2 h) + a_3 h + t \sin(a_3 h) - c_3) e^{t \cos(a_1 h)} \sin(-a_1(x-1/2 h) + t \sin(a_1 h) - c_1) e^{t \cos(a_2 h)} \sin(a_2(x-1/2 h) + a_2 h - t \sin(a_2 h) + c_2) - e^{t \cos(a_3 h)} \sin(-a_3(x-1/2 h) + t \sin(a_3 h) - c_3) e^{t \cos(a_2 h)} \sin(-a_2(x-1/2 h) + a_2 h + t \sin(a_2 h) - c_2) e^{t \cos(a_1 h)} \sin(a_1(x-1/2 h) + a_1 h - t \sin(a_1 h) + c_1) + e^{t \cos(a_3 h)} \sin(-a_3(x-1/2 h) + t \sin(a_3 h) - c_3) e^{t \cos(a_1 h)} \sin(-a_1(x-1/2 h) + a_1 h + t \sin(a_1 h) - c_1) e^{t \cos(a_2 h)} \sin(a_2(x-1/2 h) + a_2 h - t \sin(a_2 h) + c_2) + e^{t \cos(a_3 h)} \sin(a_3(x-1/2 h) + a_3 h - t \sin(a_3 h) + c_3) e^{t \cos(a_2 h)} \sin(-a_2(x-1/2 h) + a_2 h + t \sin(a_2 h) - c_2) e^{t \cos(a_1 h)} \sin(-a_1(x-1/2 h) + t \sin(a_1 h) - c_1) - e^{t \cos(a_3 h)} \sin(a_3(x-1/2 h) + a_3 h - t \sin(a_3 h) + c_3) e^{t \cos(a_1 h)} \sin(-a_1(x-1/2 h) + a_1 h + t \sin(a_1 h) - c_1) e^{t \cos(a_2 h)} \sin(-a_2(x-1/2 h) + t \sin(a_2 h) - c_2)$$

$$d_{\phi_3}(x, t) = e^{t \cos(a_3 h)} \sin(-a_3(x+1/2 h) + a_3 h + t \sin(a_3 h) - c_3) e^{t \cos(a_2 h)} \sin(-a_2(x+1/2 h) + t \sin(a_2 h) - c_2) e^{t \cos(a_1 h)} \sin(a_1(x+1/2 h) + a_1 h - t \sin(a_1 h) + c_1) - e^{t \cos(a_3 h)} \sin(-a_3(x+1/2 h) + a_3 h + t \sin(a_3 h) - c_3) e^{t \cos(a_1 h)} \sin(-a_1(x+1/2 h) + t \sin(a_1 h) - c_1) e^{t \cos(a_2 h)} \sin(a_2(x+1/2 h) + a_2 h - t \sin(a_2 h) + c_2) - e^{t \cos(a_3 h)} \sin(-a_3(x+1/2 h) + t \sin(a_3 h) - c_3) e^{t \cos(a_2 h)} \sin(-a_2(x+1/2 h) + a_2 h + t \sin(a_2 h) - c_2) e^{t \cos(a_1 h)} \sin(a_1(x+1/2 h) + a_1 h - t \sin(a_1 h) + c_1) + e^{t \cos(a_3 h)} \sin(-a_3(x+1/2 h) + t \sin(a_3 h) - c_3) e^{t \cos(a_1 h)} \sin(-a_1(x+1/2 h) + a_1 h + t \sin(a_1 h) - c_1) e^{t \cos(a_2 h)} \sin(a_2(x+1/2 h) + a_2 h - t \sin(a_2 h) + c_2) + e^{t \cos(a_3 h)} \sin(a_3(x+1/2 h) + a_3 h - t \sin(a_3 h) + c_3) e^{t \cos(a_2 h)} \sin(-a_2(x+1/2 h) + a_2 h + t \sin(a_2 h) - c_2)$$

$$1/2 h) + a_2 h + t \sin(a_2 h) - c_2) e^{t \cos(a_1 h)} \sin(-a_1(x + 1/2 h) + t \sin(a_1 h) - c_1) - e^{t \cos(a_3 h)} \sin(a_3(x + 1/2 h) + a_3 h - t \sin(a_3 h) + c_3) e^{t \cos(a_1 h)} \sin(-a_1(x + 1/2 h) + a_1 h + t \sin(a_1 h) - c_1) e^{t \cos(a_2 h)} \sin(-a_2(x + 1/2 h) + t \sin(a_2 h) - c_2)$$

We could go on and give explicit solutions for higher orders, but the expressions of these solutions would become too long for this paper. For more clarity, we only give the case of order 4 in the appendix.

Remark 2.1 *We can choose others types of generating functions to apply multiple Darboux transformation. For example, we can take :*

$$\begin{aligned} f_j(x, t) &= e^{t \cos a_j h} \cos(a_j x - t \sin(a_j h) + c_j), & \lambda_j &= 2 \cos(a_j h), \\ f_j(x, t) &= e^{t \cosh a_j h} \cosh(a_j x - t \sinh(a_j h) + c_j), & \lambda_j &= 2 \cosh(a_j h), \\ f_j(x, t) &= e^{t \cosh a_j h} \cosh(a_j x - t \sinh(a_j h) + c_j), & \lambda_j &= 2 \cosh(a_j h), \end{aligned}$$

3 Conclusion

We have given a complete description of the method of construction of solutions to the Toda equation by means of Casorati determinants or quasi Casorati determinants in the the case of the functional difference version of the Toda equation. These results were not proven in [80], and given only in the discrete case. We have constructed some explicit solutions to the this functional difference Toda equation by using the multiple Darboux transformation.

In the discrete case, we can note that some explicit solutions were proposed in particular, in [78], in [79], solutions in terms of Hankel determinants in [75].

Data Availability

The data that supports the findings of this study are available within the article.

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Appendix: Solution to the functional difference Toda equation of order 4

The function ϕ_4 defined by

$$\phi_4(x, t) = \ln(n_{\phi_4}(x, t)) - \ln(d_{\phi_4}(x, t))$$

is a solution to the Toda equation (3) with

$$\begin{aligned} n_{\phi_4}(x, t) = & e^{t \cos(a_4 h)} \sin(-a_4(x-1/2 h)+3/2 a_4 h+t \sin(a_4 h)-c_4) e^{t \cos(a_3 h)} \sin(-a_3(x- \\ & 1/2 h)+1/2 a_3 h+t \sin(a_3 h)-c_3) e^{t \cos(a_2 h)} \sin(a_2(x-1/2 h)+1/2 a_2 h-t \sin(a_2 h)+ \\ & c_2) e^{t \cos(a_1 h)} \sin(a_1(x-1/2 h)+3/2 a_1 h-t \sin(a_1 h)+c_1) - e^{t \cos(a_4 h)} \sin(-a_4(x- \\ & 1/2 h)+3/2 a_4 h+t \sin(a_4 h)-c_4) e^{t \cos(a_3 h)} \sin(-a_3(x-1/2 h)+1/2 a_3 h+t \sin(a_3 h)- \\ & c_3) e^{t \cos(a_1 h)} \sin(a_1(x-1/2 h)+1/2 a_1 h-t \sin(a_1 h)+c_1) e^{t \cos(a_2 h)} \sin(a_2(x- \\ & 1/2 h)+3/2 a_2 h-t \sin(a_2 h)+c_2) - e^{t \cos(a_4 h)} \sin(-a_4(x-1/2 h)+3/2 a_4 h+ \\ & t \sin(a_4 h)-c_4) e^{t \cos(a_3 h)} \sin(a_3(x-1/2 h)+1/2 a_3 h-t \sin(a_3 h)+c_3) e^{t \cos(a_2 h)} \sin(-a_2(x- \\ & 1/2 h)+1/2 a_2 h+t \sin(a_2 h)-c_2) e^{t \cos(a_1 h)} \sin(a_1(x-1/2 h)+3/2 a_1 h-t \sin(a_1 h)+ \\ & c_1) + e^{t \cos(a_4 h)} \sin(-a_4(x-1/2 h)+3/2 a_4 h+t \sin(a_4 h)-c_4) e^{t \cos(a_3 h)} \sin(a_3(x- \\ & 1/2 h)+1/2 a_3 h-t \sin(a_3 h)+c_3) e^{t \cos(a_1 h)} \sin(-a_1(x-1/2 h)+1/2 a_1 h+t \sin(a_1 h)- \\ & c_1) e^{t \cos(a_2 h)} \sin(a_2(x-1/2 h)+3/2 a_2 h-t \sin(a_2 h)+c_2) + e^{t \cos(a_4 h)} \sin(-a_4(x- \\ & 1/2 h)+3/2 a_4 h+t \sin(a_4 h)-c_4) e^{t \cos(a_3 h)} \sin(a_3(x-1/2 h)+3/2 a_3 h-t \sin(a_3 h)+ \end{aligned}$$

$$\begin{aligned}
& t \sin(a_4 h) + c_4) e^{t \cos(a_3 h)} \sin(-a_3(x+1/2 h) + 1/2 a_3 h + t \sin(a_3 h) - c_3) e^{t \cos(a_2 h)} \sin(-a_2(x+ \\
& 1/2 h) + 3/2 a_2 h + t \sin(a_2 h) - c_2) e^{t \cos(a_1 h)} \sin(a_1(x+1/2 h) + 3/2 a_1 h - t \sin(a_1 h) + \\
& c_1) + e^{t \cos(a_4 h)} \sin(a_4(x+1/2 h) + 1/2 a_4 h - t \sin(a_4 h) + c_4) e^{t \cos(a_3 h)} \sin(-a_3(x+ \\
& 1/2 h) + 1/2 a_3 h + t \sin(a_3 h) - c_3) e^{t \cos(a_2 h)} \sin(-a_1(x+1/2 h) + 3/2 a_1 h + t \sin(a_1 h) - \\
& c_1) e^{t \cos(a_2 h)} \sin(a_2(x+1/2 h) + 3/2 a_2 h - t \sin(a_2 h) + c_2) + e^{t \cos(a_4 h)} \sin(a_4(x+ \\
& 1/2 h) + 1/2 a_4 h - t \sin(a_4 h) + c_4) e^{t \cos(a_3 h)} \sin(a_3(x+1/2 h) + 3/2 a_3 h - t \sin(a_3 h) + \\
& c_3) e^{t \cos(a_2 h)} \sin(-a_2(x+1/2 h) + 3/2 a_2 h + t \sin(a_2 h) - c_2) e^{t \cos(a_1 h)} \sin(-a_1(x+ \\
& 1/2 h) + 1/2 a_1 h + t \sin(a_1 h) - c_1) - e^{t \cos(a_4 h)} \sin(a_4(x+1/2 h) + 1/2 a_4 h - \\
& t \sin(a_4 h) + c_4) e^{t \cos(a_3 h)} \sin(a_3(x+1/2 h) + 3/2 a_3 h - t \sin(a_3 h) + c_3) e^{t \cos(a_2 h)} \sin(-a_1(x+ \\
& 1/2 h) + 3/2 a_1 h + t \sin(a_1 h) - c_1) e^{t \cos(a_2 h)} \sin(-a_2(x+1/2 h) + 1/2 a_2 h + t \sin(a_2 h) - \\
& c_2) - e^{t \cos(a_4 h)} \sin(a_4(x+1/2 h) + 3/2 a_4 h - t \sin(a_4 h) + c_4) e^{t \cos(a_3 h)} \sin(-a_3(x+ \\
& 1/2 h) + 3/2 a_3 h + t \sin(a_3 h) - c_3) e^{t \cos(a_2 h)} \sin(-a_2(x+1/2 h) + 1/2 a_2 h + t \sin(a_2 h) - \\
& c_2) e^{t \cos(a_1 h)} \sin(a_1(x+1/2 h) + 1/2 a_1 h - t \sin(a_1 h) + c_1) + e^{t \cos(a_4 h)} \sin(a_4(x+ \\
& 1/2 h) + 3/2 a_4 h - t \sin(a_4 h) + c_4) e^{t \cos(a_3 h)} \sin(-a_3(x+1/2 h) + 3/2 a_3 h + t \sin(a_3 h) - \\
& c_3) e^{t \cos(a_2 h)} \sin(-a_1(x+1/2 h) + 1/2 a_1 h + t \sin(a_1 h) - c_1) e^{t \cos(a_2 h)} \sin(a_2(x+ \\
& 1/2 h) + 1/2 a_2 h - t \sin(a_2 h) + c_2) + e^{t \cos(a_4 h)} \sin(a_4(x+1/2 h) + 3/2 a_4 h - \\
& t \sin(a_4 h) + c_4) e^{t \cos(a_3 h)} \sin(-a_3(x+1/2 h) + 1/2 a_3 h + t \sin(a_3 h) - c_3) e^{t \cos(a_2 h)} \sin(-a_2(x+ \\
& 1/2 h) + 3/2 a_2 h + t \sin(a_2 h) - c_2) e^{t \cos(a_1 h)} \sin(a_1(x+1/2 h) + 1/2 a_1 h - t \sin(a_1 h) + \\
& c_1) - e^{t \cos(a_4 h)} \sin(a_4(x+1/2 h) + 3/2 a_4 h - t \sin(a_4 h) + c_4) e^{t \cos(a_3 h)} \sin(-a_3(x+ \\
& 1/2 h) + 1/2 a_3 h + t \sin(a_3 h) - c_3) e^{t \cos(a_2 h)} \sin(-a_1(x+1/2 h) + 3/2 a_1 h + t \sin(a_1 h) - \\
& c_1) e^{t \cos(a_2 h)} \sin(a_2(x+1/2 h) + 1/2 a_2 h - t \sin(a_2 h) + c_2) - e^{t \cos(a_4 h)} \sin(a_4(x+ \\
& 1/2 h) + 3/2 a_4 h - t \sin(a_4 h) + c_4) e^{t \cos(a_3 h)} \sin(a_3(x+1/2 h) + 1/2 a_3 h - t \sin(a_3 h) + \\
& c_3) e^{t \cos(a_2 h)} \sin(-a_2(x+1/2 h) + 3/2 a_2 h + t \sin(a_2 h) - c_2) e^{t \cos(a_1 h)} \sin(-a_1(x+ \\
& 1/2 h) + 1/2 a_1 h + t \sin(a_1 h) - c_1) + e^{t \cos(a_4 h)} \sin(a_4(x+1/2 h) + 3/2 a_4 h - \\
& t \sin(a_4 h) + c_4) e^{t \cos(a_3 h)} \sin(a_3(x+1/2 h) + 1/2 a_3 h - t \sin(a_3 h) + c_3) e^{t \cos(a_2 h)} \sin(-a_1(x+ \\
& 1/2 h) + 3/2 a_1 h + t \sin(a_1 h) - c_1) e^{t \cos(a_2 h)} \sin(-a_2(x+1/2 h) + 1/2 a_2 h + t \sin(a_2 h) - \\
& c_2)
\end{aligned}$$