Degeneracy of algebro geometric solutions of the Toda equation and Sato formulas

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Abstract

We know that the degeneracy of solutions to PDE's given in terms of theta functions on Riemann surfaces gives important results about particular solutions as in the case of the NLS equation.
Here we degenerate the so called finite gap solutions of the Toda lattice equation from the general formulation in terms of abelian functions when the gaps tends to points.
This degeneracy allows to recover the Sato formulas, without using inverse scattering theory or geometric or representation theoretic methods.

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1 Introduction

Different formulations can be chosen to represent the Toda equation. In original work of 1967 [93], the vibration of a uniform chain of particles with nonlinear interaction was studied, and Toda considered the following equation for thr n-th particle in the chain

$$m\partial_x^2(u_n) = -\partial_r(\phi)(u_n - u_{n-1}) + \partial_r(\phi)(u_{n+1} - u_n), \tag{1}$$

where m stands for the mass of the particles, $\phi(r)$ the interaction energy between adjacent particles. Some solutions were constructed for particular interactions ϕ.
The same equation was considered with $m = 1$ by Date and Tanaka in 1976 [3] and they constructed some general solutions but in terms of integrals difficult to use.
In the appendix of the paper of Dubrovin [6] published in 1981, Krichever considered the non abelian version of this equation and gave solutions in terms of theta function on Riemann surfaces.

Matveev and Stahlhoffen [91] considered in 1995 the following version of the Toda equation

$$\partial_x^2(u_n) = \exp(u_{n-1} - u_n) - \exp(u_n - u_{n+1}). \tag{2}$$

They used the Darboux transformation to contruct solutions of this equation in terms of Casoratis.
In 2014, Zhang and Zhou [95] used the following representation of the Toda equation

$$\partial_x\partial_t(u_n) = (\partial_t(u_n) + \alpha(t))\,(u_{n-1} - 2u_n + u_{n+1}). \tag{3}$$

They used the generalization of the exp-function to contruct multiwaves solutions of this equation.
More recently, Sun, Ma and Yu [95] were interessed in 2020 in the following representation of the Toda equation

$$\Delta(u_n) = 4(\exp(u_{n-1} - u_n) - \exp(u_n - u_{n+1})). \tag{4}$$

They used a logaritmic transformation to get some particular solutions in terms of logarithms.
Duarte [4] considered in 2020 the following representation of the Toda equation

$$\begin{aligned}&\alpha\partial_x^2(u(x,y,n)) + \beta\partial_y^2(u(x,y,n))\\ &= \exp(u(x,y,n-1) - u(x,y,n)) - \exp(u(x,y,n) - u(x,y,n+1)).\end{aligned}$$

He used a particular ansatz combined with the properties of Laplace's equation to contruct some solutions in terms of trigonometric functions.
Schiebold and Nilson [88] studied in 2020 another version of the Toda equation in the form

$$\partial_{xy}^2(\log(1 + u_n)) = u_{n+1} - 2u_n + u_{n+1}. \tag{5}$$

They constructed solutions in the frame of linear algebra by means of determinants.

Here, we consider the Toda equation in the following representation [1, 5, 81, 90]

$$\begin{cases} \dfrac{r_n}{r_{n+1}}\sqrt{c_n}\psi_{n+1} + v_n\psi_n + \dfrac{r_n}{r_{n-1}}\sqrt{c_{n-1}}\psi_{n-1} = \lambda\psi_n \\ \partial_t\psi_n = \dfrac{r_n}{2r_{n+1}}\sqrt{c_n}\psi_{n+1} + \partial_t(\ln r_n)\psi_n - \dfrac{r_n}{2r_{n-1}}\sqrt{c_{n-1}}\psi_{n-1}. \end{cases} \tag{6}$$

with

$$c_n = \frac{e^{2I_0}\theta^2(tV'' + nD + Z)}{\theta(tV'' + (n+1)D + Z)\theta(tV'' + (n-1)D + Z)} \tag{7}$$

$$v_n = -R_0 + \partial_t \ln\frac{\theta(tV'' + (n-1)D + Z)}{\theta(tV'' + nD + Z)}, \tag{8}$$

where θ is the classical Riemann theta function.

We know that the degeneracy of solutions of PDE's given in terms of Riemann theta functions gives some important particular solutions. In the case of the NLS equation, we have managed to construct quasi rational solution involving determinant of order N for each positive integer N depending on $2N-2$ real parameters [13]. In the case of the KdV equation we constructed from Baker Akhiezer functions, solutions in terms of Fredholm determinants and wronskians [54].
In this study, we want to recover Sato formulas for the Toda equation using this method.
From finite gap solutions given in terms of Riemann theta functions, we want to construct by degeneracy, as given in the frame of the NLS equation [79], some quasi-rational solutions and recover the Sato formulas.
Precisely, we derive multisoliton solutions from finite gap solutions by a limit transition, i.e. by making gaps tend toward points in a certain Riemann surface. This will be done in the spirit of the article [79] or more recently [54]. One strength of this approach is that it does not rely on inverse scattering theory or geometric and representation theoretic methods, which offers a fresh perspective on the problem.

We consider the Riemann surface Γ represented by $\cup_{k=1}^{g} a_k b_k a_k^{-1} b_k^{-1}$ of the algebraic curve defined by [1, 5]

$$\omega^2 = \prod_{j=1}^{2g+2} (z - E_j),$$

with $E_j \neq E_k$, $j \neq k$.

Let us consider Ω_1'' and Ω_0' [90] abelian integrals verifying

$$\int_{a_k} d\Omega_1''(P) = \int_{a_k} d\Omega_0'(P) = 0 \qquad k = 1, \ldots, g \tag{9}$$

with the following asymptotic behavior for $P = (\omega, z) \in \Gamma$ [90],

$$\Omega_1''(P) = \pm\frac{1}{2}(z + R_0 + O(z^{-1})), \qquad P \to P_\infty^{\pm} \tag{10}$$

$$\Omega_0'(P) = \pm(\ln z - I_0 + O(z^{-1})), \qquad P \to P_\infty^{\pm}, \tag{11}$$

$$\omega = \pm(z^{g+1} + O(z^g)), \qquad P \to P_\infty^{\pm} \tag{12}$$

The vectors V'', D, X and Z are defined by

$$V_j'' = \frac{1}{2\pi i} \int_{b_j} d\Omega_1'', \tag{13}$$

$$D = U(P_\infty^+) - U(P_\infty^-), \tag{14}$$

$$X = K + \sum_{j=1}^{g} U(P_j), \qquad Z = U(P_\infty^+) - X, \tag{15}$$

where K is the vector of Riemann constants.
$U(P)$ is the classical abelian integral $\int_{P_0}^{P} dU$.
The solutions of the system (6) can be written [81, 90],

$$\psi_n(P,t) = \tilde{r}_n(t) e^{n\Omega_0'(P) + t\Omega_1''(P)} \left(\frac{\theta(U(P) + tV'' + nD - X)}{\theta(U(P) - X)}\right), \tag{16}$$

with

$$\tilde{r}_n(t) = r_n(t) \left(\frac{\theta(Z)\theta(Z-D)}{\theta(tV'' + nD + Z)\theta(tV'' + (n-1)D + Z)}\right)^{\frac{1}{2}}, \tag{17}$$

2 Degeneracy of solutions

Let us suppose that E_j are real, $E_m < E_j$ if $m < j$ and try to evaluate the limits of all objects in formula (16) when E_{2m}, E_{2m-1} tends to $-\alpha_m$, $\alpha_m = \kappa_m^2$, $\kappa_m > 0$, for $1 \leq m \leq g$.
Like in the previous section, θ is constructed from the matrix of the B-periods of the surface Γ, the coefficients c_{jk} being relating with abelian differentials dU_j by

$$dU_j = \frac{\sum_{k=1}^{g} c_{jk} z^{g-k}}{\sqrt{\prod_{k=1}^{2g+2}(z - E_k)}} dz, \tag{18}$$

and coefficients c_{jk} can be obtained by solving the system of linear equations

$$\int_{a_k} dU_j = \delta_{jk}, \quad 1 \leq j \leq g, \quad 1 \leq k \leq g.$$

In the following, we will use the notations:

$$\begin{array}{ll} \kappa_{mj} = \sqrt{(\alpha_m + E_{2g+1})(\alpha_j + E_{2g+2})}, & \kappa_{jm} = \sqrt{(\alpha_j + E_{2g+1})(\alpha_m + E_{2g+2})}, \\ C_k(z) = \sqrt{(z - E_{2g+1})(\alpha_k - E_{2g+2})}, & D_k(z) = \sqrt{(z - E_{2g+2})(\alpha_k - E_{2g+1})}, \\ L_k = \sqrt{\alpha_k + E_{2g+1}}, & M_k = \sqrt{\alpha_k + E_{2g+2}}, \\ F(z) = \sqrt{z - E_{2g+1}}, & G(z) = \sqrt{z - E_{2g+2}}. \end{array} \tag{19}$$

2.1 Limit of $P(z) = \prod_{j=1}^{2g+2}(z - E_j)$

We study in this paragraph, the limit of $P(z) = \prod_{j=1}^{2g+2}(z - E_j)$.
The limit of $P(z) = \prod_{j=1}^{2g+2}(z - E_j)$ is obviously equal to

$\tilde{P}(z) = \prod_{j=1}^{g}(z+\alpha_j)^2(z-E_{2g+1})(z-E_{2g+2})$,
or with (19)

$$\tilde{P}(z) = \prod_{j=1}^{g}((z+\alpha_j)F(z)G(z))^2. \tag{20}$$

2.2 Limit of $dU_m = \dfrac{\sum_{k=1}^{g} c_{mk} z^{g-k}}{\sqrt{\prod_{k=1}^{2g+2}(z-E_k)}}dz$

In this paragraph, we study the limit of $dU_m = \dfrac{\sum_{k=1}^{g} c_{mk} z^{g-k}}{\sqrt{\prod_{k=1}^{2g+2}(z-E_k)}}dz$.
The limit of dU_m is equal to $d\tilde{U}_m = \dfrac{\varphi_m(z)}{\prod_{j=1}^{g}(z+\alpha_j)\sqrt{(z-E_{2g+1})(z-E_{2g+2})}}dz$,
where $\varphi_m(z) = \sum_{k=1}^{g} \tilde{c}_{mk} z^{g-k}$. The normalization condition takes the form in the limit

$$\int_{a_k} dU_j \to \frac{2\pi i \varphi_j(-\alpha_k)}{\prod_{m\neq k}(\alpha_m-\alpha_k)\sqrt{(\alpha_k+E_{2g+1})(\alpha_k+E_{2g+2})}} = \delta_{kj}, \tag{21}$$

which proves that the numbers $-\alpha_m$, $m \neq k$ are the zeros of the polynomials $\varphi_k(z)$, and so $\varphi_k(z)$ can be written as $\varphi_k(z) = \tilde{c}_{k1}\prod_{m\neq k}(z+\alpha_m)$.
By (21), we get in the limit

$$\tilde{c_{k1}} = \frac{\sqrt{(\alpha_k+E_{2g+1})(\alpha_k+E_{2g+2})}}{2\pi i}.$$

So

$$\varphi_k(z) = \frac{\sqrt{(\alpha_k+E_{2g+1})(\alpha_k+E_{2g+2})}}{2\pi i}\prod_{m\neq k}(z+\alpha_m).$$

Moreover

$$d\tilde{U}_k(z) = \frac{\varphi_k(z)dz}{\sqrt{(z-E_{2g+1})(z-E_{2g+2})}\prod_{m=1}^{g}(z+\alpha_m)}$$

in other words

$$d\tilde{U}_k(z) = \frac{\sqrt{(\alpha_k+E_{2g+1})(\alpha_k+E_{2g+2})}}{2\pi i\sqrt{(z-E_{2g+1})(z-E_{2g+2})}(z+\alpha_k)}dz,$$

and with (19)

$$d\tilde{U}_k(z) = \frac{L_k M_k}{2\pi i F(z)G(z)(z+\alpha_k)}dz, \tag{22}$$

2.3 Limit of B_{mk}

The subject of this subsection is the study of the limit of B_{mk}.
We have

$$I = \int_{\alpha_m}^{E_{2g+2}} dU_k \to \frac{1}{2}\tilde{B}_{mk}.$$

The integral I can be easily evaluate along the real axis on the upper sheet of surface Γ and we get

$$\tilde{B}_{mk} = \frac{i}{\pi} \ln \left| \frac{\sqrt{(\alpha_m + E_{2g+1})(\alpha_k + E_{2g+2})} + \sqrt{(\alpha_k + E_{2g+1})(\alpha_m + E_{2g+2})}}{\sqrt{(\alpha_m + E_{2g+1})(\alpha_k + E_{2g+2})} - \sqrt{(\alpha_k + E_{2g+1})(\alpha_m + E_{2g+2})}} \right|.$$

or with the previous notations (19)

$$\tilde{B}_{mk} = \frac{i}{\pi} \ln \left| \frac{\kappa_{mk} + \kappa_{km}}{\kappa_{mk} - \kappa_{km}} \right|. \tag{23}$$

So iB_{kk} tends to $-\infty$.
Moreover, we have :

$$\int_{E_{2g+2}}^{P} d\tilde{U}_k = \frac{-i}{2\pi} \ln \left| \frac{\sqrt{(z - E_{2g+1})(\alpha_k + E_{2g+2})} + \sqrt{(z - E_{2g+2})(\alpha_k + E_{2g+1})}}{\sqrt{(z - E_{2g+1})(\alpha_k + E_{2g+2})} - \sqrt{(z - E_{2g+2})(\alpha_k + E_{2g+1})}} \right|.$$

or with the previous notations (19)

$$\int_{E_{2g+2}}^{P} d\tilde{U}_k = \frac{-i}{2\pi} \ln \left| \frac{C_k(z) + D_k(z)}{C_k(z) - D_k(z)} \right|. \tag{24}$$

2.4 Limit of $\Omega_1^{''}$

In this subsection, we study the limit of $\Omega_1^{''}$.
$\Omega_1^{''}$ is an abelian integral of the second kind satisfying the conditions

$$\int_{a_k} d\Omega_1^{''}(P) = 0, \quad 1 \leq k \leq g.$$

and such that

$$\Omega_1''(P) = \pm\frac{1}{2}(z + R_0 + O(z^{-1})), \qquad P \to P_\infty^{\pm}$$

The limit of $d\Omega_1''(P)$ is equal to $d\tilde{\Omega}_1''(P) = \dfrac{\phi(z)}{\prod_{j=1}^{g}(z+\alpha_j)\sqrt{(z - E_{2g+1})(z - E_{2g+2})}} dz$,
where $\phi(z) = \sum_{k=0}^{g+1} \tilde{c}_k z^{g+1-k}$.
$\Omega_1''(P)$ satisfying the condition $\Omega_1''(P) = \pm\frac{1}{2}z + O(1)$ when $z \to \pm\infty$, we have $\tilde{c_0} = \frac{1}{2}$.
Moreover, the conditions

$$\int_{a_k} d\Omega_1''(P) = 0, \quad 1 \leq k \leq g,$$

proves that $-\alpha_1, \ldots, -\alpha_g$ are the zeros of ϕ;
thus $\phi(z) = \frac{1}{2}\prod_{m=1}^{g}(z+\alpha_m)(z-\beta)$.
We have, $P(z) = \sqrt{\prod_{k=1}^{2g+2}(z-E_k)}$ tends to
$\tilde{P}(z) = \sqrt{(z-E_{2g+1})(z-E_{2g+2})}\prod_{m=1}^{g}(z+\alpha_m)$, we get

$$d\tilde{\Omega}_1''(z) = \frac{z}{2\sqrt{(z-E_{2g+1})(z-E_{2g+2})}}dz,$$

and $\Omega_1''(P)$ tends to

$$\tilde{\Omega}_1''(P) = \int_{E_{2g+2}}^{z} \frac{u}{2\sqrt{(u-E_{2g+1})(u-E_{2g+2})}}du.$$

It can be evaluated and it gives

$$\tilde{\Omega}_1''(P) = \frac{E_{2g+1}+E_{2g+2}}{4}\ln\left|\frac{\sqrt{z-E_{2g+1}}+\sqrt{z-E_{2g+2}}}{\sqrt{z-E_{2g+1}}-\sqrt{z-E_{2g+2}}}\right| + \frac{1}{2}\sqrt{(z-E_{2g+1})(z-E_{2g+2})}.$$

With the notations defined in (19), it can be rewritten as

$$\tilde{\Omega}_1''(P) = \frac{E_{2g+1}+E_{2g+2}}{4}\ln\left|\frac{F(z)+G(z)}{F(z)-G(z)}\right| + \frac{1}{2}F(z)G(z). \qquad (25)$$

2.5 Limit of $\Omega_0^{'}$

We consider $\Omega_0^{'}$ and we study its limit.
$\Omega_0^{'}$ is an abelian integral of the third kind satisfying the conditions

$$\int_{a_k} d\Omega_0^{'}(P) = 0, \quad 1 \le k \le g.$$

and such that

$$\Omega_0'(P) = \pm(\ln z - I_0 + O(z^{-1})), \qquad P \to P_\infty^{\pm},$$

The limit of $d\Omega_0'(P)$ is equal to

$$d\tilde{\Omega'}_0(P) = \frac{\sum_{k=0}^{g}\tilde{c}_k z^{g-k}}{\prod_{j=1}^{g}(z+\alpha_j)\sqrt{(z-E_{2g+1})(z-E_{2g+2})}}dz.$$

$\Omega_0'(P)$ satisfying the condition $\Omega_0'(P) = \pm\ln(z) + O(1)$ when $z \to \pm\infty$, we have $\tilde{c_0} = 1$.
Moreover, the conditions

$$\int_{a_k} d\Omega_0'(P) = 0, \quad 1 \le k \le g,$$

prove that $-\alpha_1, \ldots, -\alpha_g$ are the zeros of ϕ;
thus $\phi(z)$ defined by $\phi(z) = \sum_{k=0}^{g}\tilde{c}_k z^{g-k}$ can be written as

$\phi(z) = \prod_{m=1}^{g}(z+\alpha_m)$.
As $P(z) = \sqrt{\prod_{k=1}^{2g+2}(z-E_k)}$ tends to
$\tilde{P}(z) = \sqrt{(z-E_{2g+1})(z-E_{2g+2})}\prod_{m=1}^{g}(z+\alpha_m)$, we get

$$d\tilde{\Omega'}_0(z) = \frac{1}{\sqrt{(z-E_{2g+1})(z-E_{2g+2})}}dz,$$

and $\Omega'_0(P)$ tends to $\tilde{\Omega'}_1(P) = \int_{E_{2g+2}}^{z} \frac{1}{\sqrt{(u-E_{2g+1})(u-E_{2g+2})}}du$.
It can be calculate and it gives

$$\tilde{\Omega}'_0(P) = \ln\left|\frac{\sqrt{z-E_{2g+1}}+\sqrt{z-E_{2g+2}}}{\sqrt{z-E_{2g+1}}-\sqrt{z-E_{2g+2}}}\right|.$$

It can be written with (19)

$$\tilde{\Omega}'_0(P) = \ln\left|\frac{F(z)+G(z)}{F(z)-G(z)}\right|. \tag{26}$$

2.6 Limit of X

We deduce the limit of X.
From the previous sections

$$\tilde{B}_{mk} = \frac{i}{\pi}\ln\left|\frac{\kappa_{mk}+\kappa_{km}}{\kappa_{mk}-\kappa_{km}}\right|,$$

$$\tilde{U}_k(P) = \frac{i}{2\pi}\ln\left|\frac{C_k(z)+D_k(z))}{C_k(z)-D_k(z)}\right|.$$

As K is defined by $K_m = \sum_{j=1}^{g} B_{jm} - \frac{m}{2}$, and $X = K + \sum_{j=1}^{g} U(P_j)$, we can write

$$X_m \to \tilde{X_m} = \frac{i}{\pi}\sum_{j=1,\, j\neq m}^{g}\ln\left|\frac{\kappa_{mj}+\kappa_{jm}}{\kappa_{mj}-\kappa_{jm}}\right| - \frac{m}{2} + \frac{i}{2\pi}\sum_{j=1}^{g}\ln\left|\frac{C_m(z_j)+D_m(z_j)}{C_m(z_j)-D_m(z_j)}\right| + i\infty. \tag{27}$$

2.7 Limit of $U(P_\infty^+)$, $U(P_\infty^-)$, D and Z

In this subsection, we study the limit of $U(P_\infty^+)$, $U(P_\infty^-)$, D and Z.
From the previous sections, it is easy to get the limits of $\tilde{U}_k$

$$\lim_{P\to P_\infty^+}\tilde{U}_k(P) = \frac{-i}{2\pi}\ln\left|\frac{\sqrt{\alpha_k+E_{2g+2}}+\sqrt{\alpha_k+E_{2g+1}}}{\sqrt{\alpha_k+E_{2g+2}}-\sqrt{\alpha_k+E_{2g+1}}}\right|,$$

or

$$\lim_{P\to P_\infty^+}\tilde{U}_k(P) = \frac{-i}{2\pi}\ln\left|\frac{M_k+L_k}{M_k+L_k}\right|. \tag{28}$$

$$D_k = U_k(P_\infty^+) - U_k(P_\infty^-) \to \tilde{D_k} = -\frac{i}{2\pi} \ln \frac{M_k + iL_k}{M_k - iL_k}. \tag{29}$$

Thus

$$\lim_{P \to P_\infty^-} \tilde{U}_k(P) = \frac{-i}{2\pi} \ln \left| \frac{\sqrt{\alpha_k + E_{2g+2}} + \sqrt{\alpha_k + E_{2g+1}}}{\sqrt{\alpha_k + E_{2g+2}} - \sqrt{\alpha_k + E_{2g+1}}} \right| - D_k$$

or

$$\lim_{P \to P_\infty^-} \tilde{U}_k(P) = \frac{-i}{2\pi} \ln \left| \frac{M_k + L_k}{M_k + L_k} \right| + \frac{i}{2\pi} \ln \left| \frac{M_k + iL_k}{M_k + iL_k} \right|. \tag{30}$$

Therefore the limit of $Z = U(P_\infty^+) - X$ is given by

$$Z_k \to \tilde{Z_k} = -\frac{i}{\pi} \sum_{j=1,\, j \neq k}^{g} \ln \left| \frac{\kappa_{kj} + \kappa_{jk}}{\kappa_{kj} - \kappa_{jk}} \right| + \frac{k}{2} - \frac{i}{2\pi} \sum_{j=1,}^{g} \ln \left| \frac{C_k(z_j) + D_k(z_j)}{C_k(z_j) - D_k(z_j)} \right| - i\infty - \frac{i}{2\pi} \ln \left| \frac{M_k + L_k}{M_k - L_k} \right| \tag{31}$$

2.8 Limit of V''

In this paragraph, the limit of V'' is studied.
$V_k'' = \frac{1}{2\pi i} \int_{b_k} d\Omega_1''$ is defined in (13).
We have $I = \int_{-\alpha_k}^{E_{2g+2}} d\Omega_1^{''} \to \frac{1}{2} \int_{b_k} d\tilde{\Omega}_1^{''}$.
As in the determination of the limit of $\Omega_1^{''}$, we have

$$\int_{-\alpha_k}^{E_{2g+2}} d\tilde{\Omega}_1''(P) = -\frac{E_{2g+1} + E_{2g+2}}{4} \ln \left| \frac{\sqrt{\alpha_k + E_{2g+1}} + \sqrt{\alpha_k + E_{2g+2}}}{\sqrt{\alpha_k + E_{2g+1}} - \sqrt{\alpha_k + E_{2g+2}}} \right|$$

$$-\frac{1}{2} \sqrt{(\alpha_k + E_{2g+1})(\alpha_k + E_{2g+2})},$$

So

$$V_k'' = \frac{1}{2\pi i} \int_{b_k} d\Omega_1'' \to \tilde{V}'' = -\frac{E_{2g+1} + E_{2g+2}}{4\pi i} \ln \left| \frac{\sqrt{\alpha_k + E_{2g+1}} + \sqrt{\alpha_k + E_{2g+2}}}{\sqrt{\alpha_k + E_{2g+1}} - \sqrt{\alpha_k + E_{2g+2}}} \right|$$

$$-\frac{1}{2\pi i} \sqrt{(\alpha_k + E_{2g+1})(\alpha_k + E_{2g+2})},$$

$$\tilde{V_k''} = \frac{E_{2g+1} + E_{2g+2}}{4\pi i} \ln \left| \frac{L_k - M_k}{L_k + M_k} \right| - \frac{1}{2\pi i} L_k M_k. \tag{32}$$

2.9 Limit of $\theta(p)$

We determine the limit of $\theta(p)$.
Let us denote A the argument of $\theta(p) = \sum_{k \in \mathbf{Z}^g} \exp\{\pi i (Bk|k) + 2\pi i (k|p)\}$.
A can be rewritten in the form

$$A = \pi i \sum_{j=1}^{g} B_{jj} k_j (k_j - 1) + 2\pi i \sum_{j>m} B_{mj} k_m k_j + \sum_{j=1}^{g} \pi i (2p_j + B_{jj}) k_j.$$

Using the inequality $k_j(k_j-1)\geq 0$ for all $k\in\mathbf{Z}^g$ and the fact that iB_{kk} tends to $-\infty$, we can reduce the limit $\tilde{\theta}$ of $\theta(p)$ to a finite sum taken over vectors $k\in\mathbf{Z}^g$ such that each k_j must be equal to 0 or 1.
In this section, we compute the limit of all the terms entering in the expression of the solution $\psi(P,t)$ in θ. We will denote p_i the arguments of these different expressions.
We first study the terms of $\tilde{r}_n$ corresponding to the arguments $p_1 = Z = U(P_\infty^+) - X$, $p_2 = Z - D$, $p_3 = tV'' + nD - Z$, $p_4 = tV'' + (n-1)D - Z$.
Then we will study the remaining terms of ψ_n corresponding to the arguments $p_5 = U(P) - X$, $p_6 = U(P) + tV'' + nD - X$.
With these notations, the solution $\psi(n,,t,z)$ can be written as

$$\psi_n(P,z) = r_n(t)\left(\frac{\theta(p_1)\theta(p_2)}{\theta(p_3)\theta(p_4)}\right)^{\frac{1}{2}}\frac{\theta(p_6)}{\theta(p_5)}e^{n\Omega_0'(P)+t\Omega_1''(P)}.$$

2.10 Limit of $\theta(p_1)$

We denote p_1 the term $Z = U(P_\infty^+) - X$.
We stufy its limit.
Then $\theta(p_1)\to\tilde{\theta}_1$:

$$\tilde{\theta}_1 = \sum_{k\in\mathbf{Z^g},\, k_j=0\ or\ 1}\exp\left\{\sum_{j>l}2\ln\left|\frac{\kappa_{lj}-\kappa_{jl}}{\kappa_{lj}+\kappa_{jl}}\right|k_jk_l + \sum_{j=1}^{g}k_j\left(\ln\left|\frac{M_j+L_j}{M_j-L_j}\right|\right.\right.$$

$$\left.\left. + \sum_{l\neq j}2\ln\left|\frac{\kappa_{jl}+\kappa_{lj}}{\kappa_{jl}-\kappa_{lj}}\right| + j\pi i + \sum_{l=1}^{g}\ln\left|\frac{C_j(z_l)+D_j(z_l)}{C_j(z_l)-D_j(z_l)}\right|\right)\right\}$$

2.11 Limit of $\theta(p_2)$

We consider $p_2 = Z - D$.We study its limit.
Then $\theta(p_2)\to\tilde{\theta}_2$: As $D_j\to\tilde{D}_j$, it is easy to see that

$$\tilde{\theta}_2 = \sum_{k\in\mathbf{Z^g},\, k_j=0\ or\ 1}\exp\left\{\sum_{j>l}2\ln\left|\frac{\kappa_{lj}-\kappa_{jl}}{\kappa_{lj}+\kappa_{jl}}\right|k_jk_l + \sum_{j=1}^{g}k_j\left(\ln\left|\frac{M_j+L_j}{M_j-L_j}\right|\right.\right.$$

$$\left.\left. + \sum_{l\neq j}2\ln\left|\frac{\kappa_{jl}+\kappa_{lj}}{\kappa_{jl}-\kappa_{lj}}\right| + j\pi i + \sum_{l=1}^{g}\ln\left|\frac{C_j(z_l)+D_j(z_l)}{C_j(z_l)-D_j(z_l)}\right| + \sum_{j=1}^{g}\ln\left|\frac{M_j+iL_j}{M_j-iL_j}\right|\right)\right\}$$

2.12 Limit of $\theta(p_3)$

Let p_3 be $tV'' + nD + Z$. We study its limit.
Then $\theta(p_3) \to \tilde{\theta}_3$:

$$\tilde{\theta}_3 = \sum_{k \in \mathbf{Z}^{\mathbf{g}},\, k_j = 0 \text{ or } 1} \exp\left\{ \sum_{j>l} 2 \ln \left| \frac{\kappa_{lj} - \kappa_{jl}}{\kappa_{lj} + \kappa_{jl}} \right| k_j k_l + \sum_{j=1}^{g} k_j \left(\ln \left| \frac{L_j + M_j}{L_j - M_j} \right| \right.\right.$$

$$+t \left(\left(\frac{E_{g+1} + E_{2g+2}}{2} \right) \ln \left| \frac{L_j - M_j}{L_j + M_j} \right| - L_j M_j \right) + n \ln \left| \frac{M_j + iL_j}{M_j - iL_j} \right|$$

$$\left.\left. + \sum_{l \neq j} 2 \ln \left| \frac{\kappa_{jl} + \kappa_{lj}}{\kappa_{jl} - \kappa_{lj}} \right| + j\pi i + \sum_{l=1}^{g} \ln \left| \frac{C_j(z_l) + D_j(z_l)}{C_j(z_l) - D_j(z_l)} \right| \right) \right\}$$

2.13 Limit of $\theta(p_4)$

We consider $p_4 = tV'' + (n-1)D + Z$; $\theta(p_4) \to \tilde{\theta}_4$. We determine its limit.
As $D_j \to \tilde{D}_j$, it is easy to verify that

$$\tilde{\theta}_4 = \sum_{k \in \mathbf{Z}^{\mathbf{g}},\, k_j = 0 \text{ or } 1} \exp\left\{ \sum_{j>l} 2 \ln \left| \frac{\kappa_{lj} - \kappa_{jl}}{\kappa_{lj} + \kappa_{jl}} \right| k_j k_l + \sum_{j=1}^{g} k_j \left(\ln \left| \frac{L_j + M_j}{L_j - M_j} \right| \right.\right.$$

$$+t \left(\left(\frac{E_{g+1} + E_{2g+2}}{2} \right) \ln \left| \frac{L_j - M_j}{L_j + M_j} \right| + L_j M_j \right) + (n-1) \ln \left| \frac{M_j + iL_j}{M_j - iL_j} \right|$$

$$\left.\left. + \sum_{l \neq j} 2 \ln \left| \frac{\kappa_{jl} + \kappa_{lj}}{\kappa_{jl} - \kappa_{lj}} \right| + j\pi i + \sum_{l=1}^{g} \ln \left| \frac{C_j(z_l) + D_j(z_l)}{C_j(z_l) - D_j(z_l)} \right| \right) \right\}$$

2.14 Limit of $\theta(p_5)$

The term $p_5 = U(P) - X$ tends to $\tilde{p}_5 = \tilde{U}(P) - \tilde{X}$.
We determine its limit.
Then $\theta(p_5) \to \tilde{\theta}_5$:

$$\tilde{\theta}_5 = \sum_{k \in \mathbf{Z}^{\mathbf{g}},\, k_j = 0 \text{ or } 1} \exp\left\{ \sum_{j>l} 2 \ln \left| \frac{\kappa_{lj} - \kappa_{jl}}{\kappa_{lj} + \kappa_{jl}} \right| k_j k_l + \sum_{j=1}^{g} k_j \left(\ln \left| \frac{C_j(z) - D_j(z)}{C_j(z) + D_j(z)} \right| \right.\right.$$

$$\left.\left. + \sum_{l \neq j} 2 \ln \left| \frac{\kappa_{jl} + \kappa_{lj}}{\kappa_{jl} - \kappa_{lj}} \right| + j\pi i + \sum_{l=1}^{g} \ln \left| \frac{C_j(z_l + D_j(z_l}{C_j(z_l - D_j(z_l} \right| \right) \right\}$$

2.15 Limit of $\theta(p_6)$

The term $p_6 = U(P) + tV'' + nD - X$ tends to $\tilde{p}_6 = \tilde{U}(P) + t\tilde{V}'' - \tilde{X} + n\tilde{D}$.
Then we determine its limit. $\theta(p_6) \to \tilde{\theta}_6$:

$$\tilde{\theta}_6 = \sum_{k\in\mathbf{Z}^{\mathbf{g}},\, k_j=0\ or\ 1} \exp\left\{ \sum_{j>l} 2\ln\left|\frac{\kappa_{lj}-\kappa_{jl}}{\kappa_{lj}+\kappa_{jl}}\right| k_j k_l + \sum_{j=1}^{g} k_j \left(\ln\left|\frac{L_j+M_j}{L_j-M_j}\right| \right.\right.$$

$$+t\left(\left(\frac{E_{g+1}+E_{2g+2}}{2}\right)\ln\left|\frac{L_j-M_j}{L_j+M_j}\right| - L_jM_j\right) + n\ln\left|\frac{M_j+iL_j}{M_j-iL_j}\right|$$

$$\left.\left. + \sum_{l\neq j} 2\ln\left|\frac{\kappa_{jl}+\kappa_{lj}}{\kappa_{jl}-\kappa_{lj}}\right| + j\pi i - \sum_{l=1}^{g}\ln\left|\frac{C_j(z_l)+D_j(z_l)}{C_j(z_l)-D_j(z_l)}\right| + \ln\left|\frac{C_j(z)-D_j(z)}{C_j(z)+D_j(z)}\right| \right) \right\}$$

2.16 Limit of solutions Ψ

From the previous section, we can give the limit $\tilde{\psi}$ of ψ. It takes the following form

$$\tilde{\psi}(z,n,t)) = \frac{r_n(t)\left(\tilde{\theta}_1\tilde{\theta}_2\right)^{\frac{1}{2}}}{\tilde{\theta}_5} \times \frac{e^{\left\{n\ln\left|\frac{F(z)+G(z)}{F(z)-G(z)}\right| + t\left(\frac{E_{2g+1}+E_{2g+2}}{4}\ln\left|\frac{F(z)+G(z)}{F(z)-G(z)}\right| + \frac{1}{2}F(z)G(z)\right)\right\}\tilde{\theta}_6}}{\left(\tilde{\theta}_3\tilde{\theta}_4\right)^{\frac{1}{2}}}$$

In the previous expression, $\tilde{\theta}_1$, $\tilde{\theta}_2$, $\tilde{\theta}_5$ are independent of n, t.
Only $\tilde{\theta}_3$, $\tilde{\theta}_4$, $\tilde{\theta}_6$ depend on n and t.
We choose the particular case in which $t = 0$.
We replace n by x.
We denote c the coefficient defined by

$$c = \frac{r_n(0)\left(\tilde{\theta}_1\tilde{\theta}_2\right)^{\frac{1}{2}}}{\tilde{\theta}_5}.$$

We denote H the function defined by

$$H(z) = \left|\frac{F(z)-G(z)}{F(z)+G(z)}\right|$$

In the different sums involving θ, only terms with $k_j = 1$ remain. So the sums can be reduced only on the subsets J of $[1;g]$ of integer $\mathbf{N}$. We denote a_j the term $a_j = -\frac{1}{2}\left|\frac{M_j+iL_j}{M_j-iL_j}\right|$. Then the term $\tilde{\theta}_6$ can be written as

$$\tilde{\theta}_6 = \sum_{J\subset\{1,\ldots,g\}} \prod_{j,l\in J,\, j<l} \ln\left|\frac{\kappa_{lj}-\kappa_{jl}}{\kappa_{lj}+\kappa_{jl}}\right|^2 \prod_{j\in J}\left|\frac{L_j+M_j}{L_j-M_j}\right|$$

$$\prod_{l\in Jl\neq j} \ln\left|\frac{\kappa_{jl}+\kappa_{lj}}{\kappa_{jl}-\kappa_{lj}}\right|^2 \prod_{j\in J}\left|\frac{C_j(z_l)+D_j(z_l)}{C_j(z_l)+D_j(z_l)}\right|\left|\frac{C_j(z)-D_j(z)}{C_j(z)+D_j(z)}\right| e^{\sum_{j\in J} j\pi i} e^{-2x\sum_{j\in J} a_j}$$

We denote τ the function defined by

$$\tau(x)=\tilde{\theta}_3(x)=\sum_{J\subset\{1,\dots,g\}}\prod_{j,l\in J,\, j<l}\ln\left|\frac{\kappa_{lj}-\kappa_{jl}}{\kappa_{lj}+\kappa_{jl}}\right|^2\prod_{j\in J}\left|\frac{L_j+M_j}{L_j-M_j}\right|$$

$$\prod_{l\in Jl\neq j} \ln\left|\frac{\kappa_{jl}+\kappa_{lj}}{\kappa_{jl}-\kappa_{lj}}\right|^2 \prod_{j\in J}\left|\frac{C_j(z_l)+D_j(z_l)}{C_j(z_l)-D_j(z_l)}\right| e^{\sum_{j\in J} j\pi i} e^{-2x\sum_{j\in J} a_j}$$

Thus, the solution $\tilde{\Psi}$ can be written

$$\tilde{\Psi}(x,z)=c\frac{\exp(xH(z))\chi(x,z)}{\sqrt{\tau(x)\tau(x-1)}}. \tag{33}$$

Here we get back tothe Sato formulation of the solution, as given in [94].
This expression is very similar to the solution expressed on page 5830, the expression of the solution is given by

$$\tilde{\Psi}(x,z)=\frac{\exp(xz)\chi(x,z)}{\sqrt{\tau(x)\tau(x-1)}}. \tag{34}$$

The difference between these two statements come from the fact that these two equations are treated differently:
Van Diejen considers the equation

$$a(x)f(x+1)+b(x)f'x)+a(x-1)f(x-1)=\lambda f(x);$$

here, we consider (**??**)

$$\frac{r_n}{r_{n+1}}\sqrt{c_n}\psi_{n+1}+v_n\psi_n+\frac{r_n}{r_{n-1}}\sqrt{c_{n-1}}\psi_{n-1}=\lambda\psi_n.$$

2.17 Limit of associated potentials

In this subsection, we determine the limit of associated potentials.
The potentials $u_n=\sqrt{c_n}$ and v_n are described in (6) and (7)

$$c_n=\frac{e^{2I_0}\theta^2(tV''+nD+Z)}{\theta(tV''+(n+1)D+Z)\theta(tV''+(n-1)D+Z)}$$

$$v_n=-R_0+\partial_t\ln\frac{\theta(tV''+(n-1)D+Z)}{\theta(tV''+nD+Z)}.$$

So the limit of u_n is equal to

$$\tilde{u_n}=k_1\frac{\tau(x)}{\sqrt{\tau(x+1)\tau(x-1)}}.$$

which is the same as in [94], (p. 5830) up to the constant k_1.
Denoting $\sigma(x) = \partial_t\theta(t\tilde{V}'' + x\tilde{D} + \tilde{Z})_{t=0}$, the limit of v_n is equal to

$$\tilde{v_n} = k_2 + \frac{\sigma(x-1)}{\tau(x-1)} - \frac{\sigma(x)}{\tau(x)}.$$

which is similar as the one written in [94], (p. 5830) up to the constant k_2.

3 Conclusion

We have used the degeneracy of solutions of some PDEs given in terms of Riemann theta functions to get some important solutions. In particular, in the case of the NLS equation, we have managed to construct quasi rational solution involving determinant of order N for each positive integer N depending on $2N2$ real parameters [13]. In the case of the KdV equation, from abekian functions, we constructed solutions in terms of Fredholm determinants and wronskians [54].
In this study, we have managed to recover Sato formulas using this method for the Toda equation. From solutions given in terms of Baker-Akiezer functions, we succeeded to construct by degeneracy, as given in the frame of the NLS equation [79], some quasi-rational solutions and recover the Sato formulas for the Toda equation.
Precisely, we derived multisoliton solutions from finite gap solutions by a limit transition, i.e. by making gaps tend toward points in a certain Riemann surface. One strength of this approach is that it does not rely on inverse scattering theory or geometric and representation theoretic methods, which offers another perspective on the problem.
The degeneracy of the solutions of the NLS equation has allowed to build new quasi-rational solutions of order N depending on $2N-2$ real parameters and their construction up to order 23, previously unknown. In the case of the KdV equation, the degeneracy of the solutions made it possible to find the solutions given by the Darboux method and thus constituted a bridge between the geometric algebra approach and the Darboux transformations framework. In this article, this is somewhat the same situation as that of the KdV equation, where we linked the algebro-geometric approach to the framework of the inverse scattering method.

Data Availability
The data that supports the findings of this study are available within the article.

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